Kinematical superalgebras

1999 J. Phys. A: Math. Gen. 32 5097

(http://iopscience.iop.org/0305-4470/32/27/309)

J. Phys. A: Math. Gen. **32** (1999) 5097–5121. Printed in the UK PII: S0305-4470(99)00613-7

Kinematical superalgebras

V Hussin†, J Negro‡ and M A del Olmo‡

† Département de Mathématiques, Université de Montréal, Montréal, Canada H3C 3J7
‡ Departamento de Física Teórica, Universidad de Valladolid E-47011 Valladolid, Spain

E-mail: jnegro@fta.uva.es and olmo@fta.uva.es and hussin@dms.umontreal.ca

Received 4 January 1999

Abstract. We investigate a class of contractions of the anti-de Sitter superalgebra in (1 + 1) and (3 + 1) space-time dimensions giving rise to the kinematical Poincaré and Galilei superalgebras. We also present faithful finite-dimensional matrix representations that are suitable for contraction in different ways.

1. Introduction

Since their introduction in 1953 [1], contraction procedures have been applied to relate Lie groups and homogeneous spaces corresponding to the various relativistic theories such as, for instance, de Sitter, Poincaré or Galilei [2]. The same formalism can be applied, with slight modifications, to many other algebraic structures such as superalgebras [3] and quantum algebras [4]. In this work we wish to use contraction methods in two respects: (i) to connect the superalgebras corresponding to the above-mentioned kinematical Lie algebras, and (ii) to define the contraction of a class of faithful matrix representations that are the most suitable to define the matrix supergroups or wave equations.

We start by paying attention to the anti-de Sitter (AdS) superalgebras corresponding to (1 + 1) and (3 + 1) space-time dimensions. Extended supergravity theories, Kaluza–Klein supergravity and general supersymmetric field theories of the Wess–Zumino type [5] can be set up in an AdS space. However, where the AdS space is most suitable is in the formulation of massless higher-spin field theories [6] that do not admit a flat space. Two-dimensional gravity models such as that of Jackiw–Teitelboim [7, 8] and dilatonic gravity have recently attracted much attention [9] since they include many interesting properties (black holes, etc) avoiding the complexity of more dimensions. These models also use an extended (1 + 1)-Poincaré superalgebra that can be obtained from an (1 + 1)-AdS superalgebra [10].

Even algebraically, (1 + 1)- and (3 + 1)-dimensional AdS superalgebras are interesting on their own. They share the property that the even sectors are isomorphic to $so(2, 1)$ and $so(3, 2)$, respectively, a coincidence that does not occur in other dimensions. This allows us to realize, in a minimal way, these superalgebras as the orthosymplectic ones $osp(1/2)$ and $osp(1/4)$, respectively. On the other hand, the dimension is more crucial in the frame of superalgebras than Lie algebras, particularly in the study of their contractions, due to the reality conditions. Therefore, besides the clarifying role of the (1 + 1)-dimensional case as a basic introduction, it contains many special features not found in (3 + 1) dimensions. Notice that in (1 + 1) dimensions the de Sitter and the anti-de Sitter superalgebras are both isomorphic, although their geometric and physical properties are quite different. However, in (3 + 1) dimensions the

0305-4470/99/275097+25$30.00

corresponding de Sitter algebras are no longer isomorphic, since (3 + 1)-deSitter $\approx so(4,1)$ and (3 + 1)-AdS $\approx so(3,2)$, but only the latter allows a Majorana representation.

This paper is organized as follows. In section 2 we write down the AdS superalgebras in (1 + 1) and (3 + 1) dimensions in terms of $osp(1/2)$ and $osp(1/4)$, respectively. We study some of the finite-dimensional irreducible representations of these superalgebras in section 3. An interesting result is the identification of these irreducible representations as nontrivial matrix subsuperalgebras of $osp(j/2k)$ with $j,k \in \boldsymbol{Z}^+$. Moreover, we present a covariant realization, in terms of gamma matrices of a particular, but physically interesting, representation of each superalgebra. In sections 4 and 5 we develop the contraction process of these superalgebras and their representations in order to obtain, first the Poincaré and secondly the Galilei superalgebras. In each case we discuss the problem of implementing an involution of the corresponding superalgebra that gives rise to the relevant grading. As a general result, we obtain a class of kinematical superalgebras from the AdS superalgebra by means of what we call 'standard contractions'. This result generalizes to the superalgebra case, the well known one that the kinematical algebras (also called inhomogeneous algebras) can be considered as contractions of a simple pseudo-orthogonal algebra (such as it is the case of AdS) [2, 11, 12]. Some remarks and conclusions end the paper.

2. AdS superalgebras

2.1. (1 + 1)-AdS superalgebra

Let us consider the Cartan basis $\{K_0, K_\pm; Q_\pm\}$ of the superalgebra $osp(1/2)$, where K_0, $K_\pm$ are the generators of the even component $sp(2,\boldsymbol{R}) \approx so(2,1)$, and $Q_\pm$ are the supercharges in the odd sector. The commutation rules are

$$\begin{aligned} &[K_0, K_\pm] = \pm K_\pm \qquad [K_+, K_-] = -2K_0 \\ &[K_0, Q_\pm] = \pm\tfrac{1}{2} Q_\pm \qquad [K_\pm, Q_\pm] = 0 \qquad [K_\pm, Q_\mp] = \mp Q_\pm \\ &\{Q_+, Q_-\} = K_0, \qquad \{Q_\pm, Q_\pm\} = K_\pm. \end{aligned} \tag{2.1}$$

The Casimir operator of $osp(1/2)$ is given by

$$\begin{aligned} \mathcal{C} &= K_0^2 - \tfrac{1}{2}(K_+K_- + K_-K_+) + \tfrac{1}{2}(Q_-Q_+ - Q_+Q_-) \\ &= \mathcal{C}_e + \tfrac{1}{2}(Q_+Q_- - Q_-Q_+) = A^2 - A/2 \end{aligned} \tag{2.2}$$

where $\mathcal{C}_e$ is the Casimir of the even part $so(2,1)$, and $A = Q_+Q_- - Q_-Q_+$ is an antisymmetric supercharge operator.

The underlying vector space of $osp(1/2)$ can be decomposed into a direct sum of three superspaces

$$osp(\tfrac{1}{2}) = \boldsymbol{n}^- \oplus \boldsymbol{h} \oplus \boldsymbol{n}^+ \tag{2.3}$$

each of them being a subsuperalgebra, such that

$$[\boldsymbol{h}, \boldsymbol{n}^\pm] \subset \boldsymbol{n}^\pm \tag{2.4}$$

where $\boldsymbol{h} = \langle K_0 \rangle$ is the Cartan subalgebra, and $\boldsymbol{n}^\pm = \langle K_\pm, Q_\pm \rangle$ are nilpotent superalgebras.

Finally, let $\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_1$ be the complex representation space for a finite-dimensional representation of $osp(1/2)$, equipped with the natural $\boldsymbol{Z}_2$ grading, i.e. 0 for the even and 1 for the odd vectors. According to [13] an inner product $\langle \cdot, \cdot \rangle$ can be defined in $\mathcal{V}$ by

$$\langle u|v\rangle = \langle u_0|v_0\rangle_0 + \langle u_1|v_1\rangle_1 \qquad u, v \in \mathcal{V} \tag{2.5}$$

where $\langle \cdot, \cdot \rangle_0$ and $\langle \cdot, \cdot \rangle_1$ are Hermitian and antiHermitian inner products of $\mathcal{V}_0$, $\mathcal{V}_1$, respectively, while $u = u_0 + u_1$, $v = v_0 + v_1$, with $u_0, v_0 \in \mathcal{V}_0$, $u_1, v_1 \in \mathcal{V}_1$. The superHermiticity of the operators acting on $\mathcal{V}$ follows directly from (2.5).

2.2. (3 + 1)-AdS superalgebra

The Cartan–Weyl basis for $osp(1/4)$ can be written in the form

$$\{H_1, H_2, E_i, F_i, (i = 1, \ldots, 4); Q_{++}, Q_{+-}, Q_{-+}, Q_{--}\}$$

where $\{H_1, H_2, E_i, F_i, (i = 1, \ldots, 4)\}$ is the Cartan–Weyl basis of the Lie algebra $sp(4, \boldsymbol{R}) \approx so(3, 2)$, that spans the even sector, and $\{Q_{++}, Q_{+-}, Q_{-+}, Q_{--}\}$ is the set of the supercharges that generate the odd sector. This is a rank-two superalgebra whose Cartan subalgebra is $\boldsymbol{h} = \langle H_1, H_2\rangle$. According to our convention the commutators are expressed in the following way (the vector $\boldsymbol{H}$ stands for the pair (H_1, H_2) and E_1, E_2 are associated to the positive simple roots of $so(3, 2)$, and similarly F_1, F_2 correspond to the negative roots):

$$\begin{array}{llll}
[\boldsymbol{H}, E_1] = (1, -1)E_1 & [\boldsymbol{H}, E_2] = (0, 1)E_2 & & \\
[\boldsymbol{H}, F_1] = (-1, 1)F_1 & [\boldsymbol{H}, F_2] = (0, -1)F_2 & & \\
[E_1, E_2] = E_3 & [E_2, E_3] = E_4 & [F_1, F_2] = -F_3 & [F_2, F_3] = -F_4 \\
[E_1, F_1] = H_1 - H_2 & [E_1, F_2] = 0 & [E_2, F_1] = 0 & [E_2, F_2] = H_2 \\
[\boldsymbol{H}, Q_{++}] = (\frac{1}{2}, \frac{1}{2})Q_{++} & [\boldsymbol{H}, Q_{+-}] = (\frac{1}{2}, -\frac{1}{2})Q_{+-} & & \\
[\boldsymbol{H}, Q_{-+}] = (-\frac{1}{2}, \frac{1}{2})Q_{-+} & [\boldsymbol{H}, Q_{--}] = (-\frac{1}{2}, -\frac{1}{2})Q_{--} & & \\
\{Q_{++}, Q_{++}\} = -2E_4 & \{Q_{++}, Q_{+-}\} = \sqrt{2}E_3 & & \\
\{Q_{++}, Q_{-+}\} = -\sqrt{2}E_2 & \{Q_{+-}, Q_{+-}\} = 2E_1 & & \\
\{Q_{--}, Q_{--}\} = 2F_4 & \{Q_{--}, Q_{-+}\} = \sqrt{2}F_3 & & \\
\{Q_{--}, Q_{+-}\} = \sqrt{2}F_2 & \{Q_{-+}, Q_{-+}\} = -2F_1 & & \\
\{Q_{++}, Q_{--}\} = -(H_1 + H_2) & \{Q_{+-}, Q_{-+}\} = H_1 - H_2. & &
\end{array} \tag{2.6}$$

The vector space of $osp(1/4)$, like $osp(1/2)$, can be expressed as the direct sum

$$osp(\tfrac{1}{4}) = \boldsymbol{n}^- \oplus \boldsymbol{h} \oplus \boldsymbol{n}^+ \tag{2.7}$$

with the Cartan subalgebra $\boldsymbol{h}$, and the positive and negative root nilpotent subsuperalgebras $\boldsymbol{n}^+ = \langle Q_{+-}, Q_{++}, \{E_i\}_{i=1}^4\rangle$ and $\boldsymbol{n}^- = \langle Q_{-+}, Q_{--}, \{F_i\}_{i=1}^4\rangle$, respectively. Quadratic and quartic Casimir operators similar to (2.2) can be given following the construction of Arnaudon *et al* [14].

3. Representations of AdS superalgebras

3.1. Irreducible finite-dimensional representations of osp(1/2)

Let $\mathcal{D}^j$ denote the $(2j + 1)$-dimensional ($(2j + 1)$D) representation of $so(2, 1)$, where j is a positive integer or half-integer. The common eigenvectors of the Casimir $\mathcal{C}_e$ and K_0, denoted by $|(j)m\rangle$, $m \in \{-j, -j + 1, \ldots, j - 1, j\}$, will be assigned to the even subspace $\mathcal{V}_0$, for example. This $\mathcal{D}^j$ representation is given by

$$\begin{aligned}
K_0|(j)m\rangle &= m|(j)m\rangle \\
K_-|(j)m\rangle &= \sqrt{(j + m)(j - m + 1)}|(j)m - 1\rangle \\
K_+|(j)m\rangle &= -\sqrt{(j - m)(j + m + 1)}|(j)m + 1\rangle
\end{aligned} \tag{3.1}$$

and

$$\mathcal{C}_e|(j)m\rangle = j(j + 1)|(j)m\rangle. \tag{3.2}$$

Note that the generator K_0 is noncompact and, when the contraction to the (1 + 1)-Poincaré algebra is performed, it is identified with the boost generator.

As the supercharges $Q_\pm$ support the $\mathcal{D}^{1/2}$ representation, the vectors $Q_\pm|(j)m\rangle$ belong to the tensor product $\mathcal{D}^j \otimes \mathcal{D}^{1/2} \approx \mathcal{D}^{j+1/2} \oplus \mathcal{D}^{j-1/2}$. In fact, they generate the carrier subspace corresponding to $\mathcal{D}^{j-1/2}$ because the vector $|(j)j\rangle$ is assumed to be the highest weight for the whole superalgebra, i.e. it is annihilated by both generators of $\boldsymbol{n}^+$, thus avoiding the $(j+1/2)$ subspace. Therefore, the representation space for the whole superalgebra is generated by the vectors

$$(K_-)^n|(j)j\rangle \qquad (K_-)^n Q_-|(j)j\rangle \qquad n \in \boldsymbol{Z}^+. \tag{3.3}$$

Now, from the commutation rules (2.1) one can check that $Q_-|(j)j\rangle \propto |(j')j'\rangle$, with $j' = j - 1/2$. So, the vectors $(K_+)^n|(j') - j'\rangle$ give rise to the $\mathcal{D}^{j-1/2}$ irreducible representation of $so(2,1)$ corresponding to the odd sector $\mathcal{V}_1$. The explicit representation of the supercharges $Q_\pm$ that can be obtained, up to a global phase factor, from the commutation rules of $osp(1/2)$ is

$$\begin{aligned} Q_+|(j)m\rangle &= -(1/\sqrt{2})\sqrt{j-m}|(j')m+\tfrac{1}{2}\rangle \\ Q_+|(j')m'\rangle &= (1/\sqrt{2})\sqrt{j'+m'+1}|(j)m'+\tfrac{1}{2}\rangle \\ Q_-|(j)m\rangle &= (1/\sqrt{2})\sqrt{j+m}|(j')m-\tfrac{1}{2}\rangle \\ Q_-|(j')m'\rangle &= (1/\sqrt{2})\sqrt{j'-m'+1}|(j)m'-\tfrac{1}{2}\rangle \end{aligned} \tag{3.4}$$

where $j' = j - 1/2$. A more compact expression of (3.4) can be given in terms of Clebsch–Gordan coefficients.

Summarizing, we have constructed a finite $(4j+1)$D representation of $osp(1/2)$, denoted by $\mathcal{T}^j$, which corresponds to the Casimir eigenvalue $\mathcal{C} = j(j+1/2)$. Hereafter, we adopt the following convention: if j (j') belongs to $\boldsymbol{Z}^+$, the states $|(j)m\rangle$ $(|(j')m'\rangle)$ will generate $\mathcal{V}_0$ while those corresponding to the half-odd positive integers $j' = j - 1/2$ $(j = j' + 1/2)$ will span $\mathcal{V}_1$. In this way $\mathcal{V}_1$ will always be considered as an even-dimensional space.

The matrices $\mathcal{T}^j(X)$ for $X \in osp(1/2)$ are easily constructed from (3.1) and (3.4). Restricting ourselves to the case $j \in \boldsymbol{Z}^+$ (the other possibility gives analogous results) we define the metric $\mathbb{K}_j$ by

$$\mathbb{K}_j = G_j \oplus J_j \tag{3.5}$$

where

$$G_j = \begin{pmatrix} & & (-1)^{2j+1} \\ & \cdots & \\ (-1) & & \end{pmatrix} \qquad J_j = \begin{pmatrix} & & (-1)^{2j'+1} \\ & \cdots & \\ (-1) & & \end{pmatrix}. \tag{3.6}$$

The matrix G_j corresponds to a pseudo-orthogonal metric with signature $(j+1, j)$ and J_j determines a symplectic metric of dimension $2j$. This means that $\mathbb{K}_j$ gives the metric for the matrix superalgebra $osp(j+1, j/2j)$:

$$\langle u|v\rangle = u^{st}\mathbb{K}_j v \qquad u, v \in \boldsymbol{R}^{4j+1}. \tag{3.7}$$

Now we can state the following theorem.

Theorem 3.1. *The irreducible representation $\mathcal{T}^j$ of $osp(1/2)$ constitutes a nontrivial matrix subsuperalgebra of $osp(j+1, j/2j)$, i.e. if $X \in osp(1/2)$ and $\mathcal{T}^j(X)$ stands for its representative matrix defined through (3.1)–(3.4), then*

$$\mathcal{T}^j(X)^{st}\mathbb{K}_j + (-1)^{\epsilon(X)}\mathbb{K}_j\mathcal{T}^j(X) = 0 \tag{3.8}$$

where $\mathbb{K}_j$ is given by (3.5) and (3.6), $\epsilon(X)$ is the grade of X, and the index, st, means supertranspose.

The proof is a simple matter of checking expression (3.8) with the matrices $\mathcal{T}^j(X)$ for all $X \in osp(1/2)$. The operators of this representation with respect to the product (3.7) are then superantiHermitian ones, i.e. $X^+ = -(-1)^{\epsilon(X)}X$.

In our notation, the fundamental matrix representation of $osp(1/2)$ is $\mathcal{T}^{1/2}$. However, the 5D representation $\mathcal{T}^1$ will be more suitable for our purposes. Indeed, the 3D even subspace $\mathcal{V}_0$ of $\mathcal{T}^1$ supports the natural representation of $so(2,1)$, so we can use this 'ambient space' to describe the physical space-time events and $so(2,1)$ can be identified with the (1 + 1)-AdS algebra. The odd sector supports the spinorial representation of $so(2,1)$. Therefore, by using this representation we are carrying, at the same time, the kinematical algebra and its fermionic description by means of the symplectic representation. According to this interpretation, $\mathcal{T}^1$ will be called the 'natural' representation of the (1 + 1)-AdS superalgebra. A basis of the carrier space $\mathcal{V}$ of $\mathcal{T}^1$ made up of $\{\mathcal{C}_e, K\}$-eigenstates is

$$\{|(1)1\rangle, |(1)0\rangle, |(1)-1\rangle, |(\tfrac{1}{2})\tfrac{1}{2}\rangle, |(\tfrac{1}{2})-\tfrac{1}{2}\rangle\}$$

so that the first three vectors generate $\mathcal{V}_0$, while the last ones, $\mathcal{V}_1$. In this basis, the matrix representation of $osp(1/2)$ is immediately computed from (3.1) and (3.4) and, from theorem 3.1, gives a nontrivial subsuperalgebra of $osp(2,1/2)$.

Having in mind the kinematical interpretation of $so(2,1)$ on $\mathcal{V}_0$, we shall diagonalize the $so(2,1)$ metric matrix G_1 of (3.6) to get

$$G = \mathrm{diag}(1,-1,1) \equiv (g_{\mu\nu}) \qquad \mu,\nu = 0,1,2, \tag{3.9}$$

which gives a new metric $\mathbb{K}$ for $osp(2,1/2)$ in the form

$$\mathbb{K} = \begin{pmatrix} 1 & 0 & 0 & * & * \\ 0 & -1 & 0 & * & * \\ 0 & 0 & 1 & * & * \\ * & * & * & 0 & 1 \\ * & * & * & -1 & 0 \end{pmatrix}. \tag{3.10}$$

The symbol $*$ stands for zero entries and is used in order to emphasize the box structure of the matrices. Next, we introduce a more geometric basis for $so(2,1)$ with the new generators $K_{\mu\nu} = -K_{\nu\mu}$ $(\mu,\nu = 0,1,2)$ defined through

$$K_0 = K_{01} \qquad K_+ = -K_{21} - K_{20} \qquad K_- = K_{21} - K_{20}. \tag{3.11}$$

The action of the generator $K_{\mu\nu}$ on the space $\mathcal{V}_0$ endowed with the metric (3.9) can be understood geometrically as a pseudo-rotation in the $\mu\nu$-plane. In this new basis, the $osp(1/2)$ commutation rules (2.1) read

$$\begin{aligned} &[K_{\mu\nu}, K_{\rho\sigma}] = g_{\mu\rho}K_{\nu\sigma} - g_{\mu\sigma}K_{\nu\rho} + g_{\nu\sigma}K_{\mu\rho} - g_{\nu\rho}K_{\mu\sigma} \\ &[K_{01}, Q_\pm] = \pm\tfrac{1}{2}Q_\pm \qquad [K_{20}, Q_\pm] = \mp\tfrac{1}{2}Q_\mp \qquad [K_{21}, Q_\pm] = \tfrac{1}{2}Q_\mp \\ &\{Q_+, Q_-\} = K_{01} \qquad \{Q_-, Q_-\} = K_{21} - K_{20} \qquad \{Q_+, Q_+\} = -K_{21} - K_{20}. \end{aligned} \tag{3.12}$$

For the corresponding matrix representation of $\mathcal{T}^1$ we use the notation

$$K_{\mu\nu} = \begin{pmatrix} M_{\mu\nu} & 0 \\ 0 & S_{\mu\nu} \end{pmatrix} \qquad Q_a = \begin{pmatrix} 0 & B_a \\ C_a & 0 \end{pmatrix} \qquad a = \pm \tag{3.13}$$

where $M_{\mu\nu}$ stands for the 3×3 vector representation of $so(2,1)$ acting on $\mathcal{V}_0$, while $S_{\mu\nu}$ denotes the 2×2 spinorial representation on $\mathcal{V}_1$. The matrix elements of $M_{\mu\nu}$ can be given explicitly with the help of the metric tensor $g_{\mu\nu}$ (3.9) as

$$(M_{\rho\sigma})^\mu{}_\nu = -g^\mu_\rho g_{\sigma\nu} + g^\mu_\sigma g_{\rho\nu}. \tag{3.14}$$

The remaining submatrices $S_{\mu\nu}$, B_a and C_a can be written with the help of γ-matrices. Indeed, let us consider the set $\{\gamma_\mu,\ \mu = 0, 1, 2\}$, associated with the (1 + 1)-AdS space-time characterized by

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}. \tag{3.15}$$

A simple choice in terms of the Pauli matrices (σ_i, $i = 1, 2, 3$) is

$$\gamma_0 = \sigma_1 \qquad \gamma_1 = \mathrm{i}\sigma_2 \qquad \gamma_2 = -\sigma_3. \tag{3.16}$$

There exists also a 'charge conjugation' matrix C, satisfying

$$\tilde{C} = -C \qquad C^+C = CC^+ = I \qquad C\tilde{\gamma}_\mu C^{-1} = -\gamma_\mu \tag{3.17}$$

where the symbol ˜ stands for the matrix transposition. For the γ given by (3.16), we can choose $C = \gamma_1$, and the matrix set $\{C, \gamma_\mu;\ \mu = 0, 1, 2\}$ constitutes a Majorana representation with real matrices.

Thus, the matrix $S_{\mu\nu}$ in (3.13) is, as usual, given by

$$S_{\mu\nu} = -S_{\nu\mu} = -\tfrac{1}{2}\gamma_\mu\gamma_\nu \qquad \mu \neq \nu \tag{3.18}$$

while the matrix components of the odd generators Q_a in (3.13) take the form

$$(B_a)^\beta{}_c = \frac{1}{\sqrt{2}}(C\gamma^\beta)_{ac} \qquad (C_a)_{c\alpha} = \frac{1}{\sqrt{2}}(\tilde{\gamma}_\alpha)_{ac} \qquad \alpha, \beta = 0, 1, 2 \qquad a, c = \pm. \tag{3.19}$$

The commutation rules (3.12) for $osp(1/2)$ can be rewritten in terms of these γ-matrices. In addition to the unchanged even commutators, we now have

$$[K_{\mu\nu}, Q_a] = Q_b(-\tfrac{1}{2}\gamma_\mu\gamma_\nu)_{ba} \qquad \{Q_a, Q_b\} = (-\tfrac{1}{2}C\gamma^\mu\gamma^\nu)_{ab}K_{\mu\nu} \tag{3.20}$$

with $\mu, \nu = 0, 1, 2$, and $a, b = +, -$.

3.2. Irreducible finite-dimensional osp(1/4) representations

First, let us recall that the even subspace of $osp(1/4)$ is the Lie algebra $so(3, 2)$ whose roots are $\{\pm\boldsymbol{e}_1, \pm\boldsymbol{e}_2, \pm\boldsymbol{e}_1 \pm \boldsymbol{e}_2\}$, where $\boldsymbol{e}_1, \boldsymbol{e}_2$ are the canonical cartesian vectors in $\boldsymbol{R}^2$ (the Cartan basis elements H_1 and H_2 are set at the origin). The simple roots are $\alpha_1 = (1, -1)$ and $\alpha_2 = (0, 1)$, and the associated fundamental weights $\lambda_1 = (1, 0)$ and $\lambda_2 = (\frac{1}{2}, \frac{1}{2})$ span the 5D and 4D fundamental representations, respectively.

Two interesting realizations of these fundamental 4D and 5D representations are given by the charges $\{Q_{++}, Q_{+-}, Q_{-+}, Q_{--}\}$, and the antisymmetric products of charges generated by $Q_{++}Q_{+-} - Q_{+-}Q_{++} \equiv Q_{++} \wedge Q_{+-}$, respectively. To show the statement for the latter case, let us first note that the quadratic products of supercharges belong to the tensor product $(0, 1)\otimes(0, 1) = (0, 2)\oplus(1, 0)\oplus(0, 0)$. In fact, $Q_{++} \wedge Q_{+-}$ is the highest weight of the (1,0) component:

$$[\boldsymbol{H}, Q_{++}\wedge Q_{+-}] = (1, 0)Q_{++}\wedge Q_{+-} \qquad [E_i, Q_{++}\wedge Q_{+-}] = 0 \qquad i = 1, 2, \ldots, 4.$$

The whole support space, that can be obtained by adding the antisymmetric products resulting from the commutators $\left[X, Q_{++} \wedge Q_{+-}\right]$, where $X \in so(3,\ 2)$, is

$$\langle Q_{++}\wedge Q_{+-}, Q_{++}\wedge Q_{-+}, Q_{+-}\wedge Q_{-+} + Q_{+-}\wedge Q_{--}, Q_{-+}\wedge Q_{--}, Q_{+-}\wedge Q_{--}\rangle. \tag{3.21}$$

The remaining $Q_{+-}\wedge Q_{-+} - Q_{++}\wedge Q_{--}$ spans the carrier space of the trivial representation $(0, 0)$. The diagrams for these fundamental representations, together with the (10D) adjoint representation are depicted in figure 1.

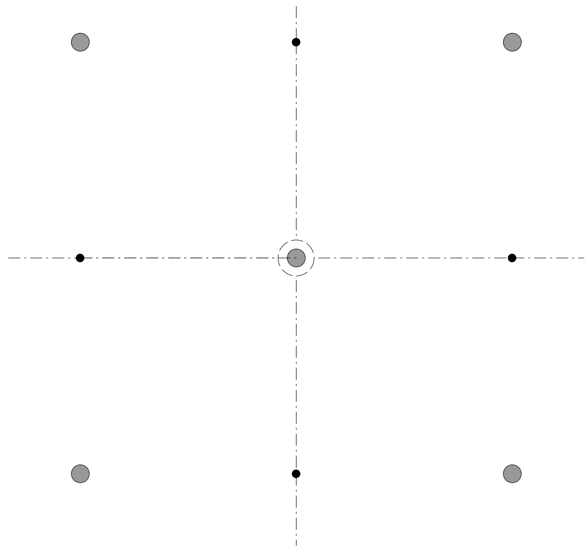

Figure 1. Diagrams for the 5D (grey circles) and 4D (black points) fundamental representations and the trivial representation (open circle).

Any other finite-dimensional irreducible representation (l_1, l_2), with $l_1, l_2 \in \boldsymbol{Z}^+$, is characterized by the highest weight $(\mu_1, \mu_2) = l_1\lambda_1 + l_2\lambda_2$. The basis vectors of these irreducible representations are denoted by $|(l_1, l_2)\alpha, \beta\rangle$, where

$$\boldsymbol{H}|(l_1, l_2)\alpha, \beta\rangle = (\alpha, \beta)|(l_1, l_2)\alpha, \beta\rangle$$

for instance, the aforementioned highest weight vector is written as $|(l_1, l_2)\mu_1, \mu_2\rangle$. Although the classification of finite-dimensional irreducible representations for the orthosymplectic superalgebras is well known [15,16], in the following we discuss some aspects of their explicit construction (the infinite-dimensional unitary representations are considered in [17, 18]).

In order to build a finite-dimensional irreducible $osp(1/4)$ representation, one starts from one (l_1, l_2) $so(3, 2)$ representation of highest weight (μ_1, μ_2), which will also be the highest weight for the whole superalgebra, i.e. it is annihilated by the elements of $\boldsymbol{n}^+$:

$$Q_{++}|(l_1, l_2)\mu_1, \mu_2\rangle = Q_{+-}|(l_1, l_2)\mu_1, \mu_2\rangle = 0 \qquad E_i|(l_1, l_2)\mu_1, \mu_2\rangle = 0 \tag{3.22}$$

with $i = 1, \ldots, 4$. The support space for the superalgebra representation is spanned by the action of the elements of $\boldsymbol{n}^-$ on the highest weight. The generators so obtained can be displayed as follows:

$$(X)^n|(l_1, l_2)\mu_1, \mu_2\rangle \tag{3.23}$$

$$(X)^n Q_{-+}|(l_1, l_2)\mu_1, \mu_2\rangle \qquad (X)^n Q_{--}|(l_1, l_2)\mu_1, \mu_2\rangle \tag{3.24}$$

$$(X)^n(Q_{-+}Q_{--} - Q_{--}Q_{-+})|(l_1, l_2)\mu_1, \mu_2\rangle \qquad n \in \boldsymbol{Z}^+ \tag{3.25}$$

where $X \in \{F_i\}_{i=1}^4$. The first set of vectors (3.23) spans the original $so(3, 2)$ representation (l_1, l_2), while those of (3.24) and (3.25) are inside the spaces $(0, 1)\otimes(l_1, l_2)$ and $(1, 0)\otimes(l_1, l_2)$, respectively.

We shall consider two relevant cases for our purposes, any other can be dealt with in the same way:

(i) $(l_1, l_2) = (0, 1)$. In this case we start with the 4D fundamental representation. The vectors inside the tensor product $(0, 1)\otimes(0, 1) = (0, 2)\oplus(1, 0)\oplus(0, 0)$ obtained by

(3.24) actually constitute the 1D trivial representation $(0, 0)$. Indeed, the highest weight vectors for the other two subrepresentations are $|(0, 2)1, 1\rangle = Q_{++}|(0, 1)\frac{1}{2}, \frac{1}{2}\rangle$ and $|(1, 0)1, 0\rangle = Q_{+-}|(0, 1)\frac{1}{2}, \frac{1}{2}\rangle$, respectively. But it is clear from (3.22), that both of them must vanish.

The second subspace (3.25) of $(1, 0)\otimes(0, 1) = (1, 1)\oplus(0, 1)$ is just the initial one, $(0, 1)$, because the $(1, 1)$ highest weight $|(1, 1)\frac{3}{2}, \frac{1}{2}\rangle = (Q_{++} \wedge Q_{+-})|(0, 1)\frac{1}{2}, \frac{1}{2}\rangle$ here, is obviously null by using (3.22). Therefore, the 5D vector space for the superalgebra representation is $\mathcal{H} = \mathcal{H}_{(0,1)} \oplus \mathcal{H}_{(0,0)}$, which corresponds to the fundamental matrix representation of $osp(1/4)$.

(ii) $(l_1, l_2) = (1, 0)$. Now, the starting point will be the second fundamental 5D representation $(1, 0)$. The first subspace (3.23) is the original $\mathcal{H}_{(1,0)}$ itself, while the second one (3.24) is included in $(1, 0) \otimes (0, 1) = (1, 1) \oplus (0, 1)$. But the representation $(1, 1)$ cannot appear here for the same reasons as in (i). For the third subspace (3.25) we must take into account the direct sum decomposition $(1, 0) \otimes (1, 0) = (2, 0) \oplus (0, 2) \oplus (0, 0)$. However, as before, one can show that the only representation allowed here is $(0, 0)$. Let us elucidate this last point with the help of the Wigner–Eckart theorem. We can write the action, in the notation explained above, as

$$Q_{++}\wedge Q_{+-}|(1, 0)1, 0\rangle = \alpha|(2, 0)2, 0\rangle + \beta|(0, 2)2, 0\rangle + \gamma|(0, 0)2, 0\rangle.$$

But the lhs vanishes (again by (3.22)) as well as the last two terms of the rhs. This implies $\alpha = 0$. In order to know more about the remaining terms, let us consider the action

$$Q_{++}\wedge Q_{-+}|(1, 0)1, 0\rangle = \beta|(0, 2)1, 1\rangle + \gamma|(0, 0)1, 1\rangle.$$

With the help of (3.22) and the commutation rules of $osp(1/4)$, we see that the lhs and the last term of the rhs actually vanish, so that necessarily $\beta = 0$. Finally, we can define

$$Q_{--}\wedge Q_{-+}|(1, 0)1, 0\rangle \equiv |(0, 0)0, 0\rangle.$$

Thus, the 10D support space for the whole superalgebra representation is

$$\mathcal{H} = (\mathcal{H}_{(1,0)} \oplus \mathcal{H}_{(0,0)}) \oplus (\mathcal{H}_{(0,1)}) \equiv \mathcal{V}_0 \oplus \mathcal{V}_1. \tag{3.26}$$

The explicit form of the 10×10 matrices representing the basis elements of $osp(1/4)$ can be obtained in a very straightforward way from the previous considerations. Nevertheless, we afford the final expressions in a more covariant way by means of the metric tensor of the $(3 + 2)$D real space underlying $so(3, 2)$, the symplectic metric of the spinor realization $sp(4, R) \approx so(3, 2)$ and a set of 4×4 γ-matrices.

Let us start by changing the $osp(1/4)$ Cartan basis of the previous section to another one given by $\{K_{\alpha\beta}, Q_a\}$, where $\alpha, \beta = 0, 1, \ldots, 4$, and $a = 1, \ldots, 4$, in a similar way as discussed in the $osp(1/2)$ case. The even elements $K_{\alpha\beta} = -K_{\beta\alpha}$ generate pseudo-rotations in the (α, β)-plane of $\boldsymbol{R}^5$ equipped with the metric $g_{\alpha\beta} = \mathrm{diag}(1, -1, -1, -1, 1)$. The points of this space will be denoted by

$$x^\alpha = (x^0, x^1, x^2, x^3, x^4) \equiv (t, \boldsymbol{x}, y) \equiv (x^\mu, y) \equiv (t, x^i, y).$$

We frequently use the following convention in what follows: the indexes α, β for $so(3, 2)$ vectors run from 0 to 4; indexes μ, ν for Lorentz $so(3, 1)$ four-vectors go from 0 to 3; indexes i, j for pure Euclidean 3D space vectors takes values from 1 to 3; and finally, indexes a, b for 4D spinors run from 1 to 4.

Recall that for $so(3, 2)$ we can define a set of matrices γ_α, $\alpha = 0, \ldots, 4$, verifying

$$\{\gamma_\alpha, \gamma_\beta\} = 2g_{\alpha\beta}. \tag{3.27}$$

Again, we can find a charge conjugation matrix $\mathcal{C}$ satisfying

$$\tilde{\mathcal{C}} = -\mathcal{C} \qquad \mathcal{C}\mathcal{C}^{+} = \mathcal{C}^{+}\mathcal{C} = I \tag{3.28}$$

which realizes the equivalence

$$\mathcal{C}\gamma_\alpha\mathcal{C}^{-1} = \tilde{\gamma}_\alpha. \tag{3.29}$$

In addition, it is possible to get a Majorana representation where all the γ_α are pure imaginary matrices while $\mathcal{C}$ is represented by a real matrix. For instance, we use the following explicit Majorana realization

$$\begin{array}{lll} \gamma_0 = \begin{pmatrix} 0 & -\sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} & \gamma_1 = \begin{pmatrix} -i\sigma_3 & 0 \\ 0 & -i\sigma_3 \end{pmatrix} & \gamma_2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \\ \gamma_3 = \begin{pmatrix} i\sigma_1 & 0 \\ 0 & i\sigma_1 \end{pmatrix} & \gamma_4 = \begin{pmatrix} -\sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} & \mathcal{C} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{array} \tag{3.30}$$

The commutation rules of $osp(1/4)$ can then be written as

$$\begin{array}{l} [K_{\mu\nu}, K_{\rho\sigma}] = g_{\mu\rho}K_{\nu\sigma} - g_{\mu\sigma}K_{\nu\rho} + g_{\nu\sigma}K_{\mu\rho} - g_{\nu\rho}K_{\mu\sigma} \\ [K_{\alpha\beta}, Q_a] = Q_b(S_{\alpha\beta})_{ba} \qquad \{Q_a, Q_b\} = (\mathcal{C}S^{\alpha\beta})_{ab}K_{\alpha\beta} \end{array} \tag{3.31}$$

where

$$S_{\alpha\beta} = -\tfrac{1}{4}[\gamma_\alpha, \gamma_\beta] \tag{3.32}$$

and the γ may be taken in the above Majorana representation.

Now, the 10D matrix representation obtained before can be expressed in the form

$$K_{\alpha\beta} = \begin{pmatrix} M_{\alpha\beta} & 0 & * \\ 0 & 0 & * \\ * & * & S_{\alpha\beta} \end{pmatrix} \qquad Q_a = \begin{pmatrix} * & * & U_a \\ * & * & v_a \\ \hat{U}_a & \hat{v}_a & * \end{pmatrix}. \tag{3.33}$$

The 5×5 matrix $M_{\alpha\beta}$ of $so(3,2)$ has elements in the δ-row and in the ϵ-column given by $(M_{\alpha\beta})^{\delta}{}_{\epsilon} = -g^{\delta}_{\alpha}g_{\beta\epsilon} + g^{\delta}_{\beta}g_{\alpha\epsilon}$. The 4×4 matrix $S_{\alpha\beta}$ is defined according to (3.32). The rectangular 5×4 (4×5) matrix U_a ($\hat{U}_a$) have elements in the α-row, b-column (b-row, α-column) determined by

$$(U_a)^{\alpha}{}_{b} = \lambda(\mathcal{C}\gamma^{\alpha})_{ab} \qquad (\hat{U}_a)_{b\alpha} = \hat{\lambda}(\tilde{\gamma}_\alpha)_{ab} \tag{3.34}$$

with $\lambda\hat{\lambda} = \frac{1}{2}$. Finally, v_a and $\hat{v}_a$ are, respectively, 1×4 and 4×1 matrices taking the form

$$(v_a)_b = \ell\mathcal{C}_{ab} \qquad (\hat{v}_a)_b = \hat{\ell}\delta_{ab} \tag{3.35}$$

with $\ell\hat{\ell} = -\frac{3}{2}$. One can show, by direct checking, that the matrices (3.33) accomplish the commutation rules (3.31). This representation has some properties that we comment upon the following.

First, notice that the charge conjugation $\mathcal{C}$ can play the role of the symplectic metric, i.e. $\tilde{S}_{\alpha\beta}\mathcal{C} + \mathcal{C}S_{\alpha\beta} = 0$ according to (3.29). So, we can define a 10×10 supermetric matrix in the form

$$\mathbb{K} = (g_{\alpha\beta}\oplus1) \oplus (k\mathcal{C}) \equiv G \oplus J \tag{3.36}$$

where k is a real parameter. By construction $\mathbb{K}$ is invariant under the action of the representation (3.33) for the even generators $K_{\alpha\beta}$. Moreover, it is also invariant under the action of the odd generators, Q_a, if $k, \lambda, \hat{\lambda}, \ell, \hat{\ell}$ are chosen properly. In fact, the equation to be satisfied by the charges is

$$(Q_a)^{st}\mathbb{K} - \mathbb{K}Q_a = 0. \tag{3.37}$$

Taking into account the form (3.33) of Q_a, equation (3.37) is transformed in two decoupled equations:

$$\lambda\mathcal{C}\gamma_\alpha - \hat{\lambda}k\tilde{\gamma}_\alpha\tilde{\mathcal{C}} = 0 \qquad \ell\mathcal{C} - \hat{\ell}k\tilde{\mathcal{C}} = 0. \tag{3.38}$$

Once we fix $k = 1$, a particular solution for the first equation is $\lambda = -\hat{\lambda} = \frac{\mathrm{i}}{\sqrt{2}}$, and for the second equation we can take $\ell = -\hat{\ell} = \sqrt{\frac{3}{2}}$. By means of this choice we get a real Majorana representation for the whole superalgebra $osp(1/4)$ which is realized as a subsuperalgebra of $osp(3, 3/4)$.

4. Contractions of the (1 + 1)-AdS superalgebra

Given a Lie superalgebra generated by $\{X_i\}$ with super-commutators,

$$[X_i, X_j]_\pm = \sum_k c_{ij}^k X_k \tag{4.1}$$

(+ stands for the anticommutators and − for the commutators) we can define a contraction by introducing a new 'rescaled' basis with the help of the contraction parameters ϵ_i, $\{X_i' = \epsilon_i X_i\}$, so that the new generators obey

$$[X_i', X_j']_\pm = \sum_k \epsilon_i\epsilon_j\epsilon_k^{-1}c_{ij}^k X_k'. \tag{4.2}$$

In the singular limit, when some of the ϵ_i go to 0, the new super-commutators may have a well defined limit originating a contracted Lie superalgebra.

An important fact is that the contraction procedure is not basis free, and according to the pursued contracted algebra one must find a suitable basis. A systematic approach choosing bases compatible with superalgebra gradings was realized by de Montigny *et al* [3, 19]. For our purposes, first we select the basis by physical or geometric considerations, which will also afford the grading relevant in the process. Secondly, we will restrict ourselves to 'continuous contractions'.

In the same way we can define the contraction of matrix representations of a given superalgebra. Let $\{M_i\}$ be the matrices representing the above superalgebra generators. We can define, by means of a nonsingular even matrix S_ϵ depending on the parameters $\{\epsilon_i\}$, the family of matrices

$$M_i(\epsilon) = \epsilon_i S_\epsilon^{-1} M_i S_\epsilon. \tag{4.3}$$

In the context of this paper, the initial matrices $\{M_i\}$ belong to an orthosymplectic superalgebra satisfying the equation

$$M_i^{st}\mathbb{K} + (-1)^{degM_i}\mathbb{K}M_i = 0 \tag{4.4}$$

where the index st stands for the supertransposition, and $\mathbb{K}$ is the metric. The redefined matrices $M_i(\epsilon)$ will verify

$$M_i(\epsilon)^{st}\mathbb{K}_\epsilon + (-1)^{degM_i(\epsilon)}\mathbb{K}_\epsilon M_i(\epsilon) = 0 \tag{4.5}$$

where

$$\mathbb{K}_\epsilon = \epsilon_{\mathbb{K}} S_\epsilon^{st}\mathbb{K}S_\epsilon \tag{4.6}$$

and $\epsilon_{\mathbb{K}}$ is an additional normalization factor depending on the contraction parameters. If in the limit $\epsilon_i \to 0$ the set $\{M_i(\epsilon), \mathbb{K}_\epsilon\}$ is well defined, we get a contraction of the matrix representation superalgebra. However, the resulting matrices may not have an orthosymplectic character since the metric matrix $\mathbb{K}_\epsilon$ could become degenerate after taking the limit. The auxiliary contraction matrix S_ϵ, that can also be dealt with using the help of the grading formalism of representations [19], is called the grading matrix. In our case it is determined by appealing to physical considerations.

4.1. *The Poincaré superalgebra as contraction of the AdS superalgebra*

In order to obtain the Poincaré superalgebra as a contraction of the AdS one, it is necessary to implement, to the odd sector, the well known contraction procedure of the even (Lie) sector (see, for instance, [20, 21]). This is a nontrivial procedure as we see below.

4.1.1. Reflection grading of AdS superalgebra. The contraction from the (anti) de Sitter to the Poincaré Lie algebra is, from a geometric point of view, a contraction around a point. In other words, the Minkowski space-time can be seen as a small neighbourhood of a point in the AdS space-time (for more details see [22]). The grading is supplied by an involution in the AdS algebra generated by a reflection leaving such a point invariant. In the following, we must implement this involution of the even sector to the whole superalgebra. The automorphism so obtained provides a $\boldsymbol{Z}_4$ grading on the superalgebra as we see below.

Let us settle the problem recalling some properties of the Lie algebras involved here. The generators of $so(2,1)$ act naturally on the ambient space $\boldsymbol{R}^3$, whose points are denoted by $(t,x,y) \equiv (x^0,x^1,x^2)$, endowed with a certain metric $g_{\mu\nu}$ of signature $(2,1)$. The metric is specified taking into account that a contraction from $so(2,1)$ to $(1+1)$ Poincaré gives rise to a flat Minkowski surface inside the ambient space parametrized by the first two coordinates $(t,x) \equiv (x^0,x^1)$. This means that for these two coordinates the metric tensor will be $(g_{ij}) = \mathrm{diag}(1,-1)$; therefore, the last (diagonal) component of $g_{\mu\nu}$ must still be fixed. In this respect,we have the following options:

(a) $(g_{\mu\nu}) = \mathrm{diag}(1,-1,1)$ and $so(2,1)$ can be interpreted as the AdS algebra.
(b) $(g_{\mu\nu}) = \mathrm{diag}(1,-1,-1)$ and we have, in fact, $so(2,1)$ as the de Sitter algebra.

As we mentioned before, both de Sitter algebras are isomorphic but their geometric and physical properties are quite different as is the case, for example, of their behaviour under contractions.

We shall consider a reflection R_y around the third axis of a cartesian coordinate system of the AdS ambient space:

$$R_y : (t,x,y) \to (-t,-x,y). \tag{4.7}$$

This reflection spans an involution, Π_y, on the even generators of pseudo-rotations

$$\Pi_y : (K_{01},K_{20},K_{21}) \to (K_{01},-K_{20},-K_{21}). \tag{4.8}$$

The action of Π_y on the charges Q_a can be implemented taking into account that, according to (3.20), they support the spinorial representation of $so(2,1)$. Thus, Π_y must be represented by γ_2 given by (3.16) up to a factor:

$$\Pi_y : (Q_+,Q_-) \to (\lambda Q_+,-\lambda Q_-). \tag{4.9}$$

The consistency with the anticommutators (3.20) fixes $\lambda = \pm i$.

We see that the grading of $osp(1/2)$, corresponding to the reflection R_y, essentially determined by the eigenvalues of Π_y, is $\boldsymbol{Z}_4$, not $\boldsymbol{Z}_2$, as one would expect from the even sector. This $\boldsymbol{Z}_4$ grading is particularly simple, since it is also compatible with that derived from the Cartan basis (2.1). Similar considerations can be developed for the other reflections: R_t and R_x. However, it is worth pointing out that, although the restrictions to the even sector commute, the corresponding implementations Π_t, Π_x and Π_y on $osp(1/2)$ are no longer commutative.

4.1.2. The (1 + 1)-Poincaré superalgebra (i). According to the grading originated by R_y the assignment of the contraction parameters is (K_0 generates the invariant subalgebra)

$$(K_0, K_\pm, Q_\pm) \to (K_0, \epsilon K_\pm, \epsilon_\pm Q_\pm). \tag{4.10}$$

We take a nontrivial option by choosing $\epsilon_\pm = \sqrt{\epsilon}$, so that in the limit $\epsilon \to 0$ we get the Poincaré superalgebra given by

$$\begin{array}{lll} [K_{01}, K_\pm] = \pm K_\pm & [K_+, K_-] = 0 & \\ [K_{01}, Q_\pm] = \pm\frac{1}{2} Q_\pm & [K_\pm, Q_\pm] = 0 & \\ \{Q_+, Q_-\} = 0 & \{Q_\pm, Q_\pm\} = K_\pm & [K_\pm, Q_\mp] = 0. \end{array} \tag{4.11}$$

Note that the only even generator keeping its spinorial character is K_{01}.

Finally, we express the commutation rules (4.11) using gamma matrices in the more physical basis (see expressions (3.12)–(3.20))

$$\begin{array}{lll} [K_{01}, K_{20}] = K_{21} & [K_{01}, K_{21}] = K_{20} & [K_{21}, K_{20}] = 0 \\ [K_{01}, Q_a] = Q_b(-\frac{1}{2}\gamma_0\gamma_1)_{ba} & [K_{2i}, Q_a] = 0 & \\ \{Q_a, Q_b\} = (-\frac{1}{2}C\gamma^2\gamma^i)_{ab} K_{2i} & i = 0, 1. & \end{array} \tag{4.12}$$

4.1.3. The natural representation of the Poincaré superalgebra. In order to find the auxiliary contraction matrix S_ϵ, it is convenient to know how the Minkowski space is obtained starting from the ambient space. Thus, let us change the AdS metric in the ambient space ($g_{\mu\nu} = \mathrm{diag}(1, -1, 1)$) in the following way

$$g_{\mu\nu}(k) = \mathrm{diag}(k^2, -k^2, 1) \qquad k \in \boldsymbol{R} \tag{4.13}$$

so that the corresponding AdS homogeneous spaces are characterized by

$$k^2x^2 - k^2t^2 + y^2 = \text{constant}. \tag{4.14}$$

Then, after applying the limit $k \to 0$, the above surface turns into the $(1 + 1)$ Minkowski space-time

$$y^2 = \text{constant}. \tag{4.15}$$

At the same time, in the whole 3D ambient space, the metric (4.13) comes into the degenerate metric $g_{\mu\nu}(0) = \mathrm{diag}(0, 0, 1)$. However, in each surface $y^2 = \text{constant}$, it is defined a surviving nondegenerate metric $(g_{ij}) = \mathrm{diag}(1, -1)$, $(i, j = 0, 1)$. With respect to the odd sector we require that along this contraction process some generators, i.e., K_{01}, preserve their spinorial behaviour. So, we propose changing the global metric (3.10) to the form

$$\mathbb{K}_\epsilon = \begin{pmatrix} -k^2 & 0 & 0 & * & * \\ 0 & k^2 & 0 & * & * \\ 0 & 0 & 1 & * & * \\ * & * & * & 0 & \eta^2 \\ * & * & * & -\eta^2 & 0 \end{pmatrix} \tag{4.16}$$

where the parameters k and η depend on ϵ in a way determined later. This allows us to choose the auxiliary contraction matrix of the vector space $\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_1$ in the form

$$S_k = \mathrm{diag}(k, k, 1, \eta, \eta). \tag{4.17}$$

Therefore, the generators in the natural representation (3.13) transform as

$$\begin{array}{ll} K'_{01} = S_\epsilon^{-1} K_{01} S_\epsilon & K'_{02} = \epsilon S_\epsilon^{-1} K_{02} S_\epsilon \\ K'_{12} = \epsilon S_\epsilon^{-1} K_{12} S_\epsilon & Q'_\pm = \sqrt{\epsilon} S_\epsilon^{-1} Q_\pm S_\epsilon. \end{array} \tag{4.18}$$

A nontrivial limit when $\epsilon \to 0$ is obtained for the values

$$k = \epsilon \qquad \eta = \sqrt{\epsilon}. \tag{4.19}$$

The contracted generator matrices are

$$K_{01} \equiv K = \begin{pmatrix} 0 & 1 & 0 & * & * \\ 1 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \\ * & * & * & 1/2 & 0 \\ * & * & * & 0 & -1/2 \end{pmatrix} \qquad K_{20} \equiv H = \begin{pmatrix} 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \\ * & * & * & 0 & 0 \\ * & * & * & 0 & 0 \end{pmatrix}$$

$$K_{21} \equiv P = \begin{pmatrix} 0 & 0 & 0 & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & * & * \\ * & * & * & 0 & 0 \\ * & * & * & 0 & 0 \end{pmatrix} \tag{4.20}$$

$$Q_+ = \frac{\sqrt{2}}{2}\begin{pmatrix} * & * & * & 1 & 0 \\ * & * & * & 1 & 0 \\ * & * & * & 0 & 0 \\ 0 & 0 & -1 & * & * \\ 0 & 0 & 0 & * & * \end{pmatrix} \qquad Q_- = \frac{\sqrt{2}}{2}\begin{pmatrix} * & * & * & 0 & -1 \\ * & * & * & 0 & 1 \\ * & * & * & 0 & 0 \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 1 & * & * \end{pmatrix}.$$

The resulting charges can be expressed in terms of the gamma matrices as

$$Q_a = \begin{pmatrix} 0 & B_a \\ C_a & 0 \end{pmatrix} \qquad a = \pm \tag{4.21}$$

where

$$(B_a)^{\beta}{}_c = \begin{cases} \dfrac{1}{\sqrt{2}}(C\gamma^i)_{ac} & \beta = i \\ 0 & \beta = 2 \end{cases} \qquad (C_a)_{c\alpha} = \begin{cases} 0 & \beta = i \\ \dfrac{1}{\sqrt{2}}(\tilde{\gamma}_2)_{ac} & \beta = 2. \end{cases} \tag{4.22}$$

It is interesting to note that the expressions (4.22) for the contracted charges Q_a can be obtained directly from (3.19) by making, in agreement to the metric change $g_{\mu\nu} = (1, -1, 1) \to g_{\mu\nu}(k) = (k^2, -k^2, 1)$, the replacements

$$\gamma_i \to k\gamma_i \qquad \gamma^i \to \frac{1}{k}\gamma^i \qquad \gamma_2 \to \gamma_2 \qquad C \to kC \qquad i = 0, 1 \tag{4.23}$$

and, afterwards, performing the limit $k \to 0$.

4.1.4. The (1 + 1) Poincaré superalgebra (ii). There are more solutions to the contraction of the (1 + 1)-AdS superalgebra corresponding to the Z_4 grading. We briefly mention the following nontrivial and nonsymmetric case by means of the assignment

$$(K_0, K_\pm, Q_+, Q_-) \to (K_0, \epsilon K_\pm, \epsilon^{1/2} Q_+, \epsilon^{3/2} Q_-). \tag{4.24}$$

Performing the limit $\epsilon \to 0$, we arrive at a second Poincaré superalgebra

$$\begin{aligned} &[K_0, K_\pm] = \pm K_\pm \qquad [K_+, K_-] = 0 \\ &[K_0, Q_\pm] = \pm\tfrac{1}{2} Q_\pm \qquad [K_\pm, Q_\pm] = [K_+, Q_-] = 0 \qquad [K_-, Q_+] = K_+ \\ &\{Q_+, Q_-\} = \{Q_-, Q_-\} = 0 \qquad \{Q_+, Q_+\} = K_+. \end{aligned} \tag{4.25}$$

In the physical basis this is

$$\begin{aligned} &[K, P] = H \qquad [K, H] = P \qquad [P, H] = 0 \\ &[K, Q_\pm] = \pm\tfrac{1}{2} Q_\pm \qquad [P, Q_+] = -[H, Q_+] = \tfrac{1}{2} Q_- \qquad [P, Q_-] = [H, Q_-] = 0 \\ &\{Q_+, Q_-\} = \{Q_-, Q_-\} = 0 \qquad \{Q_+, Q_+\} = -P - H. \end{aligned} \tag{4.26}$$

4.2. Galilei superalgebras as contractions of a Poincaré superalgebra

Now, we present the contraction procedure from the Poincaré (i) to the Galilei superalgebra. Although Poincaré (ii) superalgebra is a perfectly well defined superalgebra we do not consider it for contraction, because we are interested in obtaining the most common superalgebras in physics. However, the reader can easily obtain the contracted superalgebra of the Poincaré (ii) superalgebra following the same procedure that we are going to describe in this section.

The contraction from Poincaré (i) to the Galilei superalgebra shares, with the previous case, a similar development. For the even part this is a line-like contraction, i.e. from the geometric point of view the Galilei space-time corresponds to a neighbourhood of the Minkowski space along the temporal-axis [22]. As for Poincaré, we see in the following, that we can find more than one Galilei superalgebra. In other words, once the contraction of the even part is fixed, it does not lead to a unique contraction of the odd sector.

4.2.1. Grading of the Poincaré superalgebra. At the level of Lie algebras, choosing $\{K, H, P\}$ as the Poincaré basis, the contraction to Galilei is fulfilled by using the inversion Π_t

$$\Pi_t : (H, P, K) \to (H, -P, -K) \tag{4.27}$$

associated to a spatial inversion R_t in the Minkowski space

$$R_t : (t, x) \to (t, -x). \tag{4.28}$$

Since the only generator of the previous ones which acts nontrivially on the charge space is K (see (4.12)), we have, in principle, two basic options to implement Π_t on the odd sector:

$$\text{(a)}\quad \Pi_t = \alpha\gamma_0\gamma_2 \qquad \text{(b)}\quad \Pi_t = \alpha\gamma_0 \qquad \alpha \in \boldsymbol{C}. \tag{4.29}$$

(a) As for the first possibility, the supercharge eigenvectors of Π_t $(\alpha\gamma_0\gamma_2 Q' = \lambda Q')$ are

$$\begin{aligned} Q'_1 &= \frac{1}{\sqrt{2}}(Q_+ + iQ_-) \qquad (\lambda_1 = +i\alpha) \\ Q'_2 &= \frac{1}{\sqrt{2}}(Q_+ - iQ_-) \qquad (\lambda_2 = -i\alpha). \end{aligned} \tag{4.30}$$

In this new basis $\{Q'_1, Q'_2\}$, the commutation rules for the Poincaré superalgebra (4.12) are

$$\begin{aligned} &[K, P] = H \qquad [K, H] = P \qquad [P, H] = 0 \\ &[K, Q'_1] = \tfrac{1}{2}Q'_2 \qquad [K, Q'_2] = \tfrac{1}{2}Q'_1 \qquad [H, Q'_{1,2}] = [P, Q'_{1,2}] = 0 \\ &\{Q'_1, Q'_1\} = \{Q'_2, Q'_2\} = -P \qquad \{Q'_1, Q'_2\} = -H. \end{aligned} \tag{4.31}$$

The commutators (4.31) compel us to choose $\alpha = \pm 1$ and here we take $\alpha = 1$. This means that the relevant grading is given by $\boldsymbol{Z}_4$. If we use the physical basis (4.11) for the commutation rules with the new charges $\{Q'_1, Q'_2\}$ it is necessary to replace the set of matrices $\{\gamma_\mu, C\}$ by $\{\gamma'_\mu = \Sigma^{-1}\gamma_\mu\Sigma, C' = \tilde{\Sigma}C\Sigma\}$, where Σ is the matrix of the basis change $Q'_a = Q_b\Sigma_{ba}$. Explicitly, we have

$$\gamma'_0 = \sigma_2 \qquad \gamma'_1 = i\sigma_3 \qquad \gamma'_2 = -\sigma_1 \qquad C' = -\gamma'_1 = \sigma_2. \tag{4.32}$$

The $\boldsymbol{Z}_4$ grading allows us to choose the following rescaled basis (we drop out the prime of the charge generators since they will be used accordingly henceforth) leading to a nontrivial contraction:

$$(H, P, K, Q_1, Q_2) \to (H, \epsilon P, \epsilon K, \sqrt{\epsilon}Q_1, \sqrt{\epsilon}Q_2). \tag{4.33}$$

(b) The second option gives the supercharge eigenvectors of Π_t:

$$Q'_1 = \frac{1}{\sqrt{2}}(Q_+ + Q_-) \qquad (\lambda_1 = +\alpha) \qquad Q'_2 = \frac{1}{\sqrt{2}}(Q_+ - Q_-) \qquad (\lambda_2 = -\alpha). \tag{4.34}$$

The commutation rules for this new basis are

$$\begin{aligned} &[K, Q'_1] = \tfrac{1}{2}Q'_2 \qquad [K, Q'_2] = \tfrac{1}{2}Q'_1 \qquad [H, Q'_{1,2}] = [P, Q'_{1,2}] = 0 \\ &\{Q'_1, Q'_1\} = \{Q'_2, Q'_2\} = -H \qquad \{Q'_1, Q'_2\} = -P. \end{aligned} \tag{4.35}$$

This implies that the eigenvalues (4.34) are real and $\alpha = \pm 1$. So, the grading associated with the involution Π_t is $\boldsymbol{Z}_2$. For instance, one particular realization of Π_t on the supercharge sector when $\alpha = 1$ is:

$$\Pi_t : (Q'_1, Q'_2) \to (Q'_1, -Q'_2). \tag{4.36}$$

In this new basis $\{Q'_1, Q'_2\}$, the corresponding set of matrices $\{\gamma_\mu, C\}$ remain in the Majorana representation and are given by

$$\gamma'_0 = \sigma_3 \qquad \gamma'_1 = -i\sigma_2 \qquad \gamma'_2 = -\sigma_1 \qquad C' = -\gamma'_1 = i\sigma_2. \tag{4.37}$$

The $\boldsymbol{Z}_2$ grading leads to the following 'natural' contracting parameters (dropping the prime on the charge generators)

$$(H, P, K, Q_1, Q_2) \to (H, \epsilon P, \epsilon K, Q_1, \epsilon Q_2). \tag{4.38}$$

4.2.2. The Galilei superalgebra (i). We begin by the contraction obtained after specialization of the contraction parameters (4.33) corresponding to the $\boldsymbol{Z}_4$ grading. In the limit $\epsilon \to 0$ the supercommutators read

$$\begin{aligned} &[K, H] = P \qquad [K, P] = 0 \qquad [H, P] = 0 \qquad [K, Q_a] = 0 \\ &\{Q_a, Q_a\} = 0 \qquad \{Q_a, Q_b\} = (-C\gamma^2\gamma^1)_{ab}P. \end{aligned} \tag{4.39}$$

On the other hand, the natural matrix realization of the above superalgebra can be derived from that of Poincaré (4.20) using the auxiliary contraction matrix

$$S_\epsilon = \mathrm{diag}(1, \epsilon, 1, \sqrt{\epsilon}, \sqrt{\epsilon}). \tag{4.40}$$

The effect of (4.40) on the Minkowski space-time metric g_{ij} is to change it into a deformed metric $g_{ij}(\epsilon)$,

$$g_{ij} = \mathrm{diag}(1, -1) \to g_{ij}(\epsilon) = \mathrm{diag}(1, -\epsilon^2) \tag{4.41}$$

which in the limit $\epsilon \to 0$ provides the corresponding one for the Galilei space-time $g_{ij}(0) = \mathrm{diag}(1, 0)$. Making use of (4.40) and (4.33) in the matrix contraction (4.3), we find

$$\begin{aligned} &K = \begin{pmatrix} 0 & 0 & 0 & * & * \\ 1 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \\ * & * & * & 0 & 0 \\ * & * & * & 0 & 0 \end{pmatrix} \qquad H = \begin{pmatrix} 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \\ * & * & * & 0 & 0 \\ * & * & * & 0 & 0 \end{pmatrix} \\ &P = \begin{pmatrix} 0 & 0 & 0 & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & * & * \\ * & * & * & 0 & 0 \\ * & * & * & 0 & 0 \end{pmatrix} \\ &Q_1 = \frac{\sqrt{2}}{4}\begin{pmatrix} * & * & * & 0 & 0 \\ * & * & * & 1 & i \\ * & * & * & 0 & 0 \\ 0 & 0 & -1 & * & * \\ 0 & 0 & i & * & * \end{pmatrix} \qquad Q_2 = \frac{\sqrt{2}}{4}\begin{pmatrix} * & * & * & 0 & 0 \\ * & * & * & 1 & -i \\ * & * & * & 0 & 0 \\ 0 & 0 & -1 & * & * \\ 0 & 0 & -i & * & * \end{pmatrix}. \end{aligned} \tag{4.42}$$

The same results for the matrix realization of the supercharges can easily be obtained from those of super Poincaré (4.22) by using, in agreement with the metric contraction (4.41), the rescaling recipe:

$$\gamma_1 \to \epsilon\gamma_1 \qquad \gamma^1 \to \epsilon^{-1}\gamma^1 \qquad \gamma_\mu \to \gamma_\mu \quad (\mu \neq 1) \qquad C \to \epsilon C \tag{4.43}$$

and then taking the limit $\epsilon \to 0$.

4.2.3. The Galilei superalgebra (ii). The second way to obtain a Galilei superalgebra comes from the option (4.38) for the contracting parameters, which is related to a $\mathbf{Z}_2$ grading. Now, the Galilei supercommutators are

$$\begin{array}{llll} [K, H] = P & [K, P] = 0 & [H, P] = 0 & [K, Q_1] = \frac{1}{2}Q_2 \\ \{Q_1, Q_1\} = H & \{Q_2, Q_2\} = 0 & \{Q_1, Q_2\} = P. & \end{array} \tag{4.44}$$

A (3 + 1)D extended version of this algebra has been considered in some nonrelativistic supersymmetric field models [23]. The natural matrix representation is obtained with the help of the auxiliary contraction matrix

$$S_\epsilon = \mathrm{diag}(1, \epsilon, 1, 1, \epsilon). \tag{4.45}$$

The final resulting matrices for $\{K, P, H, Q_1, Q_2\}$ are

$$K = \begin{pmatrix} 0 & 0 & 0 & * & * \\ 1 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \\ * & * & * & 0 & 0 \\ * & * & * & \frac{1}{2} & 0 \end{pmatrix} \qquad H = \begin{pmatrix} 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \\ * & * & * & 0 & 0 \\ * & * & * & 0 & 0 \end{pmatrix}$$

$$P = \begin{pmatrix} 0 & 0 & 0 & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & * & * \\ * & * & * & 0 & 0 \\ * & * & * & 0 & 0 \end{pmatrix} \tag{4.46}$$

$$Q_1 = \frac{\sqrt{2}}{4}\begin{pmatrix} * & * & * & 1 & 0 \\ * & * & * & 0 & 1 \\ * & * & * & 0 & 0 \\ 0 & 0 & -1 & * & * \\ 0 & 0 & 0 & * & * \end{pmatrix} \qquad Q_2 = \frac{\sqrt{2}}{4}\begin{pmatrix} * & * & * & 0 & 0 \\ * & * & * & 1 & 0 \\ * & * & * & 0 & 0 \\ 0 & 0 & 0 & * & * \\ 0 & 0 & -1 & * & * \end{pmatrix}.$$

It is interesting to note, that both the commutation rules and the natural realization of this Galilei superalgebra can be obtained directly with the changes

$$C \to \begin{pmatrix} -1 & 0 \\ 0 & \epsilon^2 \end{pmatrix} \qquad \gamma_1 \to \begin{pmatrix} 0 & -\epsilon^2 \\ 1 & 0 \end{pmatrix} \qquad \gamma^1 \to \begin{pmatrix} 0 & -1 \\ \epsilon^{-2} & 0 \end{pmatrix} \tag{4.47}$$

(the other γ-matrices remain unchanged) and taking the limit $\epsilon \to 0$. In fact (4.47) gives another inequivalent realization of the gamma matrices (besides (4.43)) corresponding to the contraction metric (4.41).

4.3. General standard contraction pattern

As can be seen from the development of the previous sections, one cannot give general formulae for all the contractions of $osp(1/2)$ originating from commuting involutions of the even sector. The problem of achieving such a general setting is twofold: (i) the implementation of the involutions to the whole superalgebra is not unique, (ii) these implementations do not commute anymore. Therefore, one cannot find a fixed basis to carry out all the contractions in the cases presented here. In fact, it is necessary to change the basis in each contraction step in order to derive all the solutions.

Despite these troubles, we can get a partial answer to our proposal if we restrict ourselves to a class of contractions that will be referred to as the 'standard' contractions. They allow a common basis for all the contraction steps and are essentially determined by the involutions of the even sector.

Let us start with the supermetric $\mathbb{K}_1$ of the natural realization of $osp(1/2)$ given by (3.10). As we saw before, one way to view the contractions is as a deformation of this initial metric by means of some coefficients that, for our purposes, are chosen as follows

$$\mathbb{K}_1(\epsilon) = \begin{pmatrix} \epsilon_1^2 & 0 & 0 & * & * \\ 0 & -\epsilon_1^2\epsilon_2^2 & 0 & * & * \\ 0 & 0 & 1 & * & * \\ * & * & * & 0 & \epsilon_1\epsilon_2 \\ * & * & * & -\epsilon_1\epsilon_2 & 0 \end{pmatrix}. \tag{4.48}$$

The corresponding auxiliary contraction matrix is given by

$$S(\epsilon) = \mathrm{diag}(\epsilon_1, \epsilon_1\epsilon_2, 1, \sqrt{\epsilon_1\epsilon_2}, \sqrt{\epsilon_1\epsilon_2}). \tag{4.49}$$

Due to the change of metric, the $osp(1/2)$ generators (3.12) are also affected by the involved contraction coefficients in the following way

$$K'_{01} = \epsilon_2 K_{01} \qquad K'_{20} = \epsilon_1 K_{20} \qquad K'_{21} = \epsilon_1\epsilon_2 K_{21} \qquad Q'_{\pm} = \sqrt{\epsilon_1\epsilon_2}\, Q_{\pm}. \tag{4.50}$$

Now, the initial super commutators (3.20) become

$$\begin{aligned} &[K'_{01}, K'_{20}] = K'_{21} \qquad [K'_{01}, K'_{21}] = \epsilon_2^2 K'_{20} \qquad [K'_{20}, K'_{21}] = \epsilon_1^2 K'_{01} \\ &[K'_{20}, Q'_a] = \epsilon_1 Q'_b(-\tfrac{1}{2}\gamma_2\gamma_0)_{ba} \\ &[K'_{21}, Q'_a] = \epsilon_1\epsilon_2 Q'_b(-\tfrac{1}{2}\gamma_2\gamma_1)_{ba} \\ &[K'_{01}, Q'_a] = \epsilon_2 Q'_b(-\tfrac{1}{2}\gamma_0\gamma_1)_{ba} \\ &\{Q'_a, Q'_b\} = \epsilon_1(-\tfrac{1}{2}C\gamma^0\gamma^1)_{ab}K'_{01} + \epsilon_2(-\tfrac{1}{2}C\gamma^2\gamma^0)_{ab}K'_{20} + (-\tfrac{1}{2}C\gamma^2\gamma^1)_{ab}K'_{21}. \end{aligned} \tag{4.51}$$

The corresponding natural matrix realization of the superalgebra generators take the form

$$\begin{aligned} &K'_{01} = \begin{pmatrix} 0 & \epsilon_2^2 & 0 & * \\ 1 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ * & * & * & \epsilon_2 S_{01} \end{pmatrix} \qquad K'_{20} = \begin{pmatrix} 0 & 0 & 1 & * \\ 0 & 0 & 0 & * \\ -\epsilon_1^2 & 0 & 0 & * \\ * & * & * & \epsilon_1 S_{20} \end{pmatrix} \\ &K'_{21} = \begin{pmatrix} 0 & 0 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & \epsilon_1^2\epsilon_2^2 & 0 & * \\ * & * & * & \epsilon_1\epsilon_2 S_{21} \end{pmatrix} \qquad Q'_a = \begin{pmatrix} 0 & B'_a \\ C'_a & 0 \end{pmatrix} \qquad a = \pm \end{aligned} \tag{4.52}$$

where the submatrices $S_{\mu\nu}$ were defined in (3.18), and

$$\begin{aligned} &(B'_a)^0{}_c = \epsilon_2 \frac{1}{\sqrt{2}}(C\gamma^0)_{ac} \qquad (B'_a)^1{}_c = \frac{1}{\sqrt{2}}(C\gamma^1)_{ac} \qquad (B'_a)^2{}_c = \epsilon_1\epsilon_2 \frac{1}{\sqrt{2}}(C\gamma^2)_{ac} \\ &(C'_a)_{c0} = \epsilon_1 \frac{1}{\sqrt{2}}(\tilde{\gamma}_0)_{ac} \qquad (C'_a)_{c1} = \epsilon_1\epsilon_2 \frac{1}{\sqrt{2}}(\tilde{\gamma}_1)_{ac} \qquad (C'_a)_{c2} = \epsilon_1 \frac{1}{\sqrt{2}}(\tilde{\gamma}_2)_{ac}. \end{aligned} \tag{4.53}$$

The above expressions for the supercharges can be somewhat simplified by writing them as

$$(B'_a)^{\beta}{}_c = \frac{1}{\sqrt{2}}(C'\gamma'^{\beta})_{ac} \qquad (C'_a)_{c\alpha} = \frac{1}{\sqrt{2}}(\tilde{\gamma'}_{\alpha})_{ac} \tag{4.54}$$

where the new gamma and charge conjugation matrices are normalized according to the metric change (4.48),

$$\begin{aligned} &\gamma_0' = \epsilon_1\gamma_0 \qquad \gamma_1' = \epsilon_1\epsilon_2\gamma_0 \qquad \gamma_2' = \gamma_2 \\ &\gamma'^0 = \frac{1}{\epsilon_1}\gamma^0 \qquad \gamma'^1 = \frac{1}{\epsilon_1\epsilon_2}\gamma^1 \qquad C' = \epsilon_1\epsilon_2 C. \end{aligned} \tag{4.55}$$

Although some of the limits when $\epsilon_i \to 0$ of the expressions (4.55) are not well defined, the above terms appear conveniently mixed in the natural realization (4.54) avoiding any divergence problem.

As far as the deformed metric does not change any sign of the initial metric signature (i.e., provided $\epsilon_i \geqslant 0$) we have well defined superalgebras even in the limit $\epsilon_i \to 0$. For instance, we can first consider $\epsilon_1 \to 0, \epsilon_2 \to 1$ to get the (1 + 1) Poincaré (i) superalgebra, and after take $\epsilon_2 \to 0$ to end with Galilei (i). But we can also begin with $\epsilon_1 \to 1, \epsilon_2 \to 0$ to find a (1 + 1) Newton–Hooke superalgebra to arrive at Galilei (i) following a different route.

However, changing the sign of the initial metric means that some ϵ_i become pure imaginary complex numbers, and consequently our final superalgebra has complex structure constants. If one wants to get real superalgebras associated with a different signature it is necessary to start from the corresponding Majorana representation that in some cases, depending on the dimension, must be doubled.

5. Contractions of the (3 + 1)-AdS superalgebra

We now address the contraction of $osp(1/4)$ identified as the (3+1)-AdS superalgebra. As was discussed for the case of (1 + 1) dimensions, in the contraction to Poincaré, we must examine the grading corresponding to the reflection around the y-axis, R_y. This is implemented here in the odd sector by γ_4 up to a factor. Hence, the odd sector basis adequate for the contraction process is composed of eigenvectors of γ_4, that is, by choosing a sort of 'chiral' representation for the γ_α matrices.

On the other hand, we are restricted to a Majorana representation in order to get real structure constants. However, both conditions cannot be fulfilled at the same time in (3 + 1) dimensions, so, if we want to keep the reality condition, then we are just led to the class of 'standard' contractions defined in the previous section. The other possibility, that will not be discussed here, is to double the dimension of the odd sector. We shall also afford, in the last section, a nonstandard (complex) contraction for a chiral representation since we find it instructive.

The same problem of incompatibility of grading and reality appears when one tries to get the Galilei superalgebra from the Poincaré one. Therefore, we display in the next section, the general form of the standard contractions that include these two important superalgebras (here, we do not give more general and compact formulae for all standard contractions, since it is out of our present scope). Later we comment on particular details concerning the Poincaré and Galilei superalgebras.

5.1. Standard contractions

Here we translate the results of the standard contractions for the $(1+1)$ dimensions with some slight and direct changes. The initial metric is given by the 10×10 matrix

$$\mathbb{K} = \begin{pmatrix} 1 & 0 & 0 & * & * \\ 0 & -I_3 & 0 & * & * \\ 0 & 0 & 1 & * & * \\ * & * & * & 1 & * \\ * & * & * & * & \mathcal{C} \end{pmatrix} \tag{5.1}$$

where I_3 stands for the 3D unit matrix and $\mathcal{C}$ is the charge in the Majorana representation. The modified metric has the form

$$\mathbb{K}(\epsilon) = \begin{pmatrix} \epsilon_1^2 & 0 & 0 & * & * \\ 0 & -\epsilon_1^2\epsilon_2^2 I_3 & 0 & * & * \\ 0 & 0 & 1 & * & * \\ * & * & * & \tau^2 & * \\ * & * & * & * & \epsilon_1\epsilon_2\mathcal{C} \end{pmatrix} \tag{5.2}$$

corresponding to the grading matrix

$$S(\epsilon) = \begin{pmatrix} \epsilon_1 & 0 & 0 & * & * \\ 0 & \epsilon_1\epsilon_2 I_3 & 0 & * & * \\ 0 & 0 & 1 & * & * \\ * & * & * & \tau & * \\ * & * & * & * & \sqrt{\epsilon_1\epsilon_2} I_4 \end{pmatrix} \tag{5.3}$$

where τ is an ϵ-dependent parameter not specified yet. The generators are affected by the coefficients ϵ_1, ϵ_2 in the following way:

$$K'_{0i} = \epsilon_2 K_{0i} \qquad K'_{40} = \epsilon_1 K_{40} \qquad K'_{ij} = K_{ij} \qquad K'_{4i} = \epsilon_1\epsilon_2 K_{4i} \qquad Q'_a = \sqrt{\epsilon_1\epsilon_2} Q_a.$$

Thus, according to these changes, the initial super commutators (3.31) come into

$$\begin{aligned} &[K'_{0i}, K'_{40}] = K'_{4i} \qquad [K'_{0i}, K'_{4i}] = \epsilon_2^2 K'_{40} \qquad [K'_{0i}, K'_{0j}] = \epsilon_2^2 K'_{ij} \qquad [K'_{0i}, K'_{ij}] = K'_{0j} \\ &[K'_{40}, K'_{4i}] = \epsilon_1^2 K'_{0i} \qquad [K'_{4i}, K'_{ij}] = K'_{4j} \qquad [K'_{ij}, K'_{jk}] = K'_{ik} \\ &[K'_{40}, Q'_a] = \epsilon_1 Q'_b(-\tfrac{1}{2}\gamma_4\gamma_0)_{ba} \qquad [K'_{4i}, Q'_a] = \epsilon_1\epsilon_2 Q'_b(-\tfrac{1}{2}\gamma_4\gamma_i)_{ba} \\ &[K'_{0i}, Q'_a] = \epsilon_2 Q'_b(-\tfrac{1}{2}\gamma_0\gamma_i)_{ba} \qquad [K'_{ij}, Q'_a] = Q'_b(-\tfrac{1}{2}\gamma_i\gamma_j)_{ba} \\ &\{Q'_a, Q'_b\} = \epsilon_1(-\tfrac{1}{2}C\gamma^0\gamma^i)_{ab}K'_{0i} + \epsilon_2(-\tfrac{1}{2}C\gamma^4\gamma^0)_{ab}K'_{40} + (-\tfrac{1}{2}C\gamma^4\gamma^i)_{ab}K'_{4i} \\ &\qquad\qquad +\epsilon_1\epsilon_2(-\tfrac{1}{2}C\gamma^i\gamma^j)_{ab}K'_{ij}. \end{aligned} \tag{5.4}$$

The natural 10×10 matrix realizations corresponding to the above family of superalgebras for the even generators take the form

$$\begin{aligned} K'_{0i} &= \begin{pmatrix} 0 & \epsilon_2^2 \boldsymbol{e}_i & 0 & * & * \\ \boldsymbol{e}_i^t & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \\ * & * & * & 0 & * \\ * & * & * & * & \epsilon_2 S_{01} \end{pmatrix} \qquad K'_{ij} = \begin{pmatrix} 0 & * & * & * & * \\ * & \boldsymbol{J}_{ij} & * & * & * \\ * & * & 0 & * & * \\ * & * & * & 0 & * \\ * & * & * & * & S_{ij} \end{pmatrix} \\ K'_{40} &= \begin{pmatrix} 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & * & * \\ -\epsilon_1^2 & 0 & 0 & * & * \\ * & * & * & 0 & * \\ * & * & * & * & \epsilon_1 S_{40} \end{pmatrix} \qquad K'_{4i} = \begin{pmatrix} 0 & 0 & 0 & * & * \\ 0 & 0 & \boldsymbol{e}_i^t & * & * \\ 0 & \epsilon_1^2\epsilon_2^2 \boldsymbol{e}_i & 0 & * & * \\ * & * & * & 0 & * \\ * & * & * & * & \epsilon_1\epsilon_2 S_{4i} \end{pmatrix}. \end{aligned} \tag{5.5}$$

Here, $\boldsymbol{e}_i$ denotes the ith row vector of the canonical $\boldsymbol{R}^3$ basis, and $\boldsymbol{J}_{ij}$ the $so(3)$ matrix generators. The odd sector is represented by

$$Q'_a = \begin{pmatrix} * & * & U'_a \\ * & * & v'_a \\ \hat{U}'_a & \hat{v}'_a & * \end{pmatrix} \qquad a = 1, \ldots, 4 \tag{5.6}$$

where

$$(U'_a)^{\alpha}{}_b = \lambda(\mathcal{C}'\gamma'^{\alpha})_{ab} \qquad (\hat{U}'_a)_{b\alpha} = \hat{\lambda}(\tilde{\gamma}'_\alpha)_{ab} \qquad (v_a)'_b = \ell'\mathcal{C}'_{ab} \qquad (\hat{v}_a)'_b = \hat{\ell}'\delta_{ab} \tag{5.7}$$

with $\lambda\hat{\lambda} = \frac{1}{2}$, $\ell'\hat{\ell}' = -\frac{3}{2}$.

The new gamma and charge matrices are rescaled having in mind the metric change,

$$\begin{aligned} &\gamma'_0 = \epsilon_1\gamma_0 \qquad \gamma'_i = \epsilon_1\epsilon_2\gamma_i \qquad \gamma'_4 = \gamma_4 \\ &\gamma'^0 = \frac{1}{\epsilon_1}\gamma^0 \qquad \gamma'^i = \frac{1}{\epsilon_1\epsilon_2}\gamma^i \qquad C' = \epsilon_1\epsilon_2 C. \end{aligned} \tag{5.8}$$

After substituting (5.8) in (5.7) we get well behaved expressions, even in the limit $\epsilon_i \to 0$.

5.2. (3 + 1)-Poincaré superalgebra as contraction from the AdS superalgebra

Taking the limit $\epsilon_1 \to 0$ and $\epsilon_2 \to 1$ in (5.4) we have the Poincaré superalgebra commutation rules (we omit the primes in the generators of the contracted superalgebra)

$$\begin{aligned} &[K_{\mu\nu}, K_{\rho\sigma}] = g_{\nu\sigma}K_{\mu\rho} + g_{\mu\rho}K_{\nu\sigma} - g_{\mu\sigma}K_{\nu\rho} - g_{\nu\rho}K_{\mu\sigma} \\ &[K_{\mu\nu}, P_\rho] = g_{\mu\rho}P_\nu - g_{\nu\rho}P_\mu \\ &[K_{\mu\nu}, Q_a] = Q_b(-\tfrac{1}{2}\gamma_\mu\gamma_\nu)_{ba} \qquad \mu \neq \nu \\ &[P_\mu, Q_a] = 0 \\ &\{Q_a, Q_b\} = (\hat{\mathcal{C}}\gamma^\mu)_{ab}P_\mu \end{aligned} \tag{5.9}$$

where we have adopted the usual convention $P_\mu \equiv K_{4\mu}$ for the translation generators of Poincaré, and we have redefined the charge conjugation as $\hat{\mathcal{C}} \equiv \gamma_4\mathcal{C}$.

In order to get a better understanding of the natural representation for this superalgebra we begin with the contraction of the supercharges. The general expression for Q_a was given in (5.6). After contraction it becomes

$$(U_a)^\beta{}_c = \begin{cases} \dfrac{1}{\sqrt{2}}(\mathcal{C}\gamma^\mu)_{ac} & \beta = \mu \\ 0 & \beta = 4 \end{cases} \qquad (\hat{U}_a)_{c\alpha} = \begin{cases} 0 & \alpha = \mu \\ \dfrac{1}{\sqrt{2}}(\tilde{\gamma}_4)_{ac} & \alpha = 4. \end{cases} \tag{5.10}$$

For the row v_a and column $\hat{v}_a$ there are two basic options depending on the entry τ of the grading matrix:

(i) If $\tau = \sqrt{\epsilon_1}$ it leads us to

$$v_a = \hat{v}_a = 0. \tag{5.11}$$

(ii) If we choose $\tau = 1$ or $\tau = \epsilon_1$, we get

$$v_a = 0 \qquad (\hat{v}_a)_b = \hat{\ell}\delta_{ab} \qquad \text{or} \qquad (v_a)_b = \ell\mathcal{C}_{ab} \qquad \hat{v}_a = 0. \tag{5.12}$$

Case (i) gives a matrix representation of the whole Poincaré superalgebra which is a direct sum of an irreducible nontrivial 9D representation plus the trivial 1D representation. Case (ii) gives rise to a 10D reducible triangular representation (not completely reducible) that includes (going to the quotient) the previous one. Hereafter, we shall restrict ourselves to the more elementary 9D irreducible representation that will be referred to as the 'natural' super Poincaré representation. Thus, the odd generators are represented by the 9D matrices

$$Q_a = \begin{pmatrix} * & U_a \\ \hat{U}_a & * \end{pmatrix} \qquad a = 1, \ldots, 4 \tag{5.13}$$

with U_a, $\hat{U}_a$ defined by (5.10). Finally, the even generators are schematically represented as

$$K_{\mu\nu} = \begin{pmatrix} M_{\mu\nu} & * & * \\ * & 0 & * \\ * & * & S_{\mu\nu} \end{pmatrix} \qquad P_\mu = \begin{pmatrix} 0 & \boldsymbol{e}^t_\mu & * \\ 0 & 0 & * \\ * & * & 0 \end{pmatrix} \tag{5.14}$$

where $M_{\mu\nu}$ corresponds to the defining 4D matrix representation of the Lorentz algebra $so(3,1)$, $\boldsymbol{e}^t_\mu$ is a 4D column vector, and $S_{\mu\nu}$ denotes the $so(3,1)$-spinorial representation. Let us mention that by means of an equivalence we can get rid of γ_4 in (5.10) without affecting (5.14):

$$(U_a)^\beta{}_c = \begin{cases} \dfrac{-1}{\sqrt{2}}(\hat{\mathcal{C}}\gamma^\mu)_{ac} & \beta = \mu \\ 0 & \beta = 4 \end{cases} \qquad (\hat{U}_a)_{c\alpha} = \begin{cases} 0 & \alpha = \mu \\ \dfrac{1}{\sqrt{2}}\delta_{ac} & \alpha = 4 \end{cases} \tag{5.15}$$

where $\hat{\mathcal{C}}$ is the same Poincaré charge conjugation that appears in the commutation rules (5.9). This is actually the matrix form presented in Cornwell's book [24].

5.3. (3 + 1)-Galilei and Newton–Hooke superalgebras

The Galilei superalgebra can be obtained through the contraction $\epsilon_1 \to 0, \epsilon_2 \to 0$. The standard Galilei generators are taken as follows: $J_i \equiv \epsilon_{ijk} J_{jk}$ (ϵ_{ijk} is the totally skewsymmetric rank-three tensor), $K_i \equiv K_{0i}$, $H \equiv K_{40}$, $P_i \equiv K_{4i}$. This superalgebra is characterized by the supercommutators

$$\begin{aligned} &[\boldsymbol{J}, \boldsymbol{J}] = -\boldsymbol{J} \qquad [\boldsymbol{J}, \boldsymbol{P}] = -\boldsymbol{P} \qquad [\boldsymbol{J}, \boldsymbol{K}] = -\boldsymbol{K} \\ &[\boldsymbol{K}, H] = \boldsymbol{P} \qquad [K_i, P_j] = [P_i, H] = 0 \qquad i, j = 1, 2, 3 \\ &[H, Q_a] = [\boldsymbol{P}, Q_a] = 0 \qquad [\boldsymbol{J}, Q_a] = Q_b(\boldsymbol{S})_{ba} \\ &\{Q_a, Q_b\} = (\hat{\mathcal{C}}\gamma^i)_{ab} P_i \end{aligned} \tag{5.16}$$

where $[\boldsymbol{A}, \boldsymbol{B}] = \boldsymbol{C}$ stands for $[A_i, B_j] = \epsilon_{ijk} C_k$ $(i, j, k = 1, 2, 3)$, $[\boldsymbol{A}, B] = \boldsymbol{C}$ for $[A_i, B] = C_i$ $(i = 1, 2, 3)$, and $S_i = \epsilon_{ijk} S_{jk}$ are the spinor matrices. The natural representation for the even generators is

$$K_i = \begin{pmatrix} 0 & 0 & 0 & * \\ \boldsymbol{e}_i & 0 & 0 & * \\ 0 & 0 & 0 & * \\ * & * & * & 0 \end{pmatrix} \qquad J_i = \begin{pmatrix} 0 & * & * & * \\ * & \boldsymbol{J}_i & * & * \\ * & * & 0 & * \\ * & * & * & \boldsymbol{S}_i \end{pmatrix}$$
$$H = \begin{pmatrix} 0 & 0 & 1 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ * & * & * & 0 \end{pmatrix} \qquad P_i = \begin{pmatrix} 0 & 0 & 0 & * \\ 0 & 0 & \boldsymbol{e}_i & * \\ 0 & 0 & 0 & * \\ * & * & * & 0 \end{pmatrix} \tag{5.17}$$

and the charges

$$Q_a = \begin{pmatrix} * & U_a \\ \hat{U}_a & * \end{pmatrix} \qquad a = 1, \ldots, 4 \tag{5.18}$$

are given in terms of the submatrices

$$(U_a)^\beta{}_c = \begin{cases} \dfrac{-1}{\sqrt{2}}(\hat{\mathcal{C}}\gamma^i)_{ac} & \beta = i \\ 0 & \beta = 0, 4 \end{cases} \qquad (\hat{U}_a)_{c\alpha} = \begin{cases} 0 & \alpha = \mu \\ \dfrac{1}{\sqrt{2}}\delta_{ac} & \alpha = 4. \end{cases} \tag{5.19}$$

The (oscillating) Newton–Hooke superalgebra is obtained by taking $\epsilon_1 \to 1$ and $\epsilon_2 \to 0$. The corresponding commutators are

$$\begin{aligned}
&[\boldsymbol{J},\boldsymbol{J}] = -\boldsymbol{J} \qquad [\boldsymbol{J},\boldsymbol{P}] = -\boldsymbol{P} \qquad [\boldsymbol{J},\boldsymbol{K}] = -\boldsymbol{K}\\
&[\boldsymbol{K},H] = \boldsymbol{P} \qquad [K_i,P_j] = 0 \qquad [H,\boldsymbol{P}] = \boldsymbol{K}\\
&[\boldsymbol{J},Q_a] = Q_b(\boldsymbol{S})_{ba} \qquad [\boldsymbol{H},Q_a] = Q_b(-\tfrac{1}{2}\gamma_4\gamma_0)_{ba}\\
&\{Q_a,Q_b\} = (-\mathcal{C}\gamma^0\gamma^i)_{ab}K_i + (\hat{\mathcal{C}}\gamma^i)_{ab}P_i.
\end{aligned} \tag{5.20}$$

The natural matrix realization is the same as for super Galilei except for the time translation generator that is now represented by

$$H = \begin{pmatrix} 0 & 0 & 1 & * \\ 0 & 0 & 0 & * \\ -1 & 0 & 0 & * \\ * & * & * & S_{40} \end{pmatrix}. \tag{5.21}$$

5.4. A nonstandard (3 + 1) Poincaré superalgebra

Here we discuss a nonstandard contraction of the AdS superalgebra that gives a Poincaré superalgebra having complex structure constants. In this respect we make use of a chiral representation where γ_4 is diagonal and is also called the 'modified chiral representation' [24]:

$$\begin{aligned}
&\gamma_0 = \begin{pmatrix} 0 & -\sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \qquad \gamma_i = \begin{pmatrix} 0 & -\sigma_i\sigma_2 \\ \sigma_2\sigma_i & 0 \end{pmatrix}\\
&\gamma_4 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad \mathcal{C} = \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}.
\end{aligned}$$

We can use a more compact notation with a set of four sigma matrices: $\sigma_\mu = (I,\sigma_i)$, $\overline{\sigma}_\mu = (I,-\sigma_i)$, so that

$$\gamma_\mu = \begin{pmatrix} 0 & -\sigma_\mu\sigma_2 \\ -\sigma_2\overline{\sigma}_\mu & 0 \end{pmatrix}. \tag{5.22}$$

With the help of the above expressions we can write the spinor representation of $so(3,2)$ as

$$S_{\mu\nu} = \begin{pmatrix} \Gamma^L_{\mu\nu} & 0 \\ 0 & \Gamma^R_{\mu\nu} \end{pmatrix} \equiv \frac{1}{2}\begin{pmatrix} \sigma_\mu\sigma_\nu & 0 \\ 0 & (\sigma_\mu\sigma_\nu)^t \end{pmatrix} \qquad S_{4\mu} = \frac{1}{2}\begin{pmatrix} 0 & \sigma_\mu\sigma_2 \\ -\sigma_2\overline{\sigma}_\mu & 0 \end{pmatrix}. \tag{5.23}$$

Now, in order to implement the contraction we fix the grading of the supercharges according to the eigenvalues of γ_4:

Q_A, $(A = 1, 2)$ corresponding to eigenvalue +1, it will be assigned grade 0
$Q_{\overline{A}}$, $(\overline{A} = 1, 2)$, of eigenvalue -1 and grade 1.

Therefore, we have a $\boldsymbol{Z}_2$ grading with the following contraction parameters:

$$K_{\mu\nu} \to K_{\mu\nu} \qquad K_{4\mu} \to \epsilon K_{4\mu} \qquad Q_A \to Q_A \qquad Q_{\overline{A}} \to \epsilon Q_{\overline{A}}. \tag{5.24}$$

The supercommutation rules obtained in the limit $\epsilon \to 0$ gives a nonstandard Poincaré superalgebra:

$$\begin{aligned}
&[K_{\mu\nu},K_{\rho\sigma}] = g_{\nu\sigma}K_{\mu\rho} + g_{\mu\rho}K_{\nu\sigma} - g_{\mu\sigma}K_{\nu\rho} - g_{\nu\rho}K_{\mu\sigma}\\
&[K_{\mu\nu},P_\rho] = g_{\mu\rho}P_\nu - g_{\nu\rho}P_\mu\\
&[K_{\mu\nu},Q_A] = Q_B(\Gamma^L_{\mu\nu})_{BA} \qquad [K_{\mu\nu},Q_{\overline{A}}] = Q_{\overline{B}}(\Gamma^R_{\mu\nu})_{BA}\\
&[P_\mu,Q_{\overline{A}}] = 0 \qquad [P_\mu,Q_A] = Q_{\overline{B}}(-\tfrac{1}{2}\sigma_2\overline{\sigma}_\mu)_{BA}\\
&\{Q_{\overline{A}},Q_{\overline{B}}\} = 0 \qquad \{Q_A,Q_{\overline{B}}\} = (\overline{\sigma}^\mu)^*_{AB}P_\mu \qquad \{Q_A,Q_B\} = (\sigma_2\Gamma^{L\,\mu\nu})_{AB}K_{\mu\nu}.
\end{aligned} \tag{5.25}$$

The natural matrix representation for this superalgebra can be derived by means of an appropriate grading matrix, in agreement with the grading of the subspace of charges:

$$S(\epsilon) = \mathrm{diag}(\epsilon, \epsilon I_3, 1, 1, I_2, \epsilon I_2). \tag{5.26}$$

Hence, for the even generators, we have

$$K_{\mu\nu} = \begin{pmatrix} M_{\mu\nu} & * & * \\ * & 0 & * \\ * & * & S_{\mu\nu} \end{pmatrix} \qquad P_\mu = \begin{pmatrix} 0 & \boldsymbol{e}_\mu^t & * \\ 0 & 0 & * \\ * & * & S_\mu \end{pmatrix} \tag{5.27}$$

with

$$S_{\mu\nu} = \begin{pmatrix} \Gamma_{\mu\nu}^L & 0 \\ 0 & \Gamma_{\mu\nu}^R \end{pmatrix} \qquad S_\mu = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ -\sigma_2\overline{\sigma}_\mu & 0 \end{pmatrix}. \tag{5.28}$$

The supercharge matrices have the general structure shown in (5.6) and (5.7), but in this case the γ and $\mathcal{C}'$ matrices take a different form:

$$\begin{aligned} \gamma'_\mu &= \begin{pmatrix} 0 & -\epsilon^2\sigma_\mu\sigma_2 \\ -\sigma_2\overline{\sigma}_\mu & 0 \end{pmatrix} \qquad \gamma'_4 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \\ \gamma^{\mu\prime} &= \begin{pmatrix} 0 & -\sigma^\mu\sigma_2 \\ -\epsilon^{-2}\sigma_2\overline{\sigma}^\mu & 0 \end{pmatrix} \qquad \mathcal{C}' = \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\epsilon^2\sigma_2 \end{pmatrix}. \end{aligned} \tag{5.29}$$

Substituting the above expressions in (5.6) and (5.7) we have well defined limits when $\epsilon \to 0$.

Finally, we want to comment, briefly, how in this framework the Poincaré algebra is described by a faithful representation obtained as a contraction of the symplectic matrices of $sp(4, \boldsymbol{R})$. The corresponding representation matrices given in (5.28) leave the contracted metric invariant

$$\mathcal{C} = \begin{pmatrix} \mathrm{i}\sigma_2 & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.30}$$

The space-time points are put in correspondence with the 4D matrices

$$(t, \boldsymbol{x}) = (x^\mu) \to x^\mu\gamma_\mu + \gamma_4 \equiv X \tag{5.31}$$

with (putting $\epsilon = 0$ in (5.29))

$$\gamma_\mu = \begin{pmatrix} 0 & 0 \\ -\sigma_2\overline{\sigma}_\mu & 0 \end{pmatrix}. \tag{5.32}$$

Hence, the action of the Poincaré generators $A \in \{S_{\mu\nu}, S_\mu\}$ given by (5.28) on the space-time is

$$A : X \to [A, X], \tag{5.33}$$

that comes from the group action

$$\mathrm{e}^{\lambda A} : X \to \mathrm{e}^{\lambda A} X \mathrm{e}^{-\lambda X}. \tag{5.34}$$

This action, leads to the transformation of $x_\mu\overline{\sigma}^\mu$ by means of the Lorentz representation $D^{(0,1/2)}$. A similar contraction would lead to the transformation of $x_\mu\sigma^\mu$ under $D^{(1/2,0)}$.

6. Conclusions

In this work we have dealt with the problem of physical contractions of the AdS superalgebras in (1+1) and (3+1) dimensions. By physical contractions we mean those including in the even sector the well known contractions of the AdS Lie algebra originating kinematical algebras, for example, the Poincaré and Galilei Lie algebras. An interesting point is the nonuniqueness of the solutions to the above problem, related to the existence of different gradings providing the contraction, and the importance that the dimension has in this respect. However, we have been able to give a contraction formalism, referred to as the 'standard contractions', having very general properties. We studied, in detail, the grading relevant in the process, how to get the appropriate basis and the reality conditions characterizing our contractions.

Along the same lines, we also considered the contraction of the natural representations. These are faithful matrix representations that play a similar role to the defining matrix representations of $so(2,1)$ and $so(3,2)$ which provide the ambient space for the AdS Lie algebras in $(1+1)$ and $(3+1)$ dimensions. Such natural representations are readily built out of the metric tensor and the class of gamma matrices associated with the corresponding algebra. This list of properties is not accomplished, for instance, in the fundamental representations of $osp(1/2)$ or $osp(1/4)$ (corresponding to both AdS superalgebras) because they cannot supply us with faithful representations for all the kinematical contractions such as the Poincaré and Galilei superalgebras.

There are several directions in which this investigation can be complemented. Immediate work is in progress to enlarge this theory to the superalgebra extensions [25, 26] where we expect new features especially in $(1+1)$ dimensions.

Acknowledgments

VH would like to thank the DGICYT of the Ministerio de Educación y Cultura de España for supporting his sabbatical stay (grant SAB95-0196) during which part of this work was completed. This work was partially supported by DGES of the Ministerio de Educación y Cultura of Spain under Project PB95-0719, the Junta de Castilla y León (Spain), NSERC of Canada and FCAR du Québec.

References

[1] Inónú E and Wigner E P 1953 *Proc. Natl Acad. Sci. USA* **39** 510
[2] Bacry H and Lévy-Leblond J M 1968 *J. Math. Phys.* **8** 1605
[3] de Montigny M and Patera J 1991 *J. Phys. A: Math. Gen.* **21** 525
[4] Ballesteros A, Herranz F J and Negro J 1997 *J. Phys. A: Math. Gen.* **30** 6797
[5] Keck B W 1975 *J. Phys. A: Math. Gen.* **8** 1819
[6] Vasiliev M A 1990 *Phys. Lett.* B **243** 378
Vasiliev M A 1987 *Fortschr. Phys.* **35** 741
[7] Teitelboim C 1983 *Phys. Lett.* B **126** 41
[8] Jackiw R 1985 *Nucl. Phys. Lett.* B **252** 343
[9] Cangemi D and Leblanc M 1994 *Nucl. Phys.* B **420** 363
[10] Rivelles V O 1994 *Preprint* IFUSP-P/1025
[11] de Montigny M, Patera J and Tolar J 1994 *J. Math. Phys.* **35** 405
[12] Herranz F J, de Montigny M, del Olmo M A and Santander M 1994 *J. Phys. A: Math. Gen.* **27** 2515
[13] Gradechi A M and Nieto L M 1996 *Commun. Math. Phys.* **175** 521
[14] Arnaudon D, Bauer M and Frappat L 1997 *Commun. Math. Phys.* **187** 429
[15] Kac V G 1978 *Representations of Classical Lie Superalgebras (Lecture Notes in Mathematics vol 676)* (Berlin: Springer)
[16] Scheunert M 1979 *The Theory of Lie Superalgebras (Lecture Notes in Mathematics vol 716)* (Berlin: Springer)
[17] Heidereich H 1982 *Phys. Lett.* B **110** 461
[18] Nicolai H 1984 Representations of supersymmetry in anti-de Sitter space *Supersymmetry and Supergravity'84* ed B de Wit, P Fayet and P van Nieuwenhuizen (Singapore: World Scientific)
[19] Moody R V and Patera J 1991 *J. Phys. A: Math. Gen.* **21** 2227
[20] Gazeau J P and Hussin V 1995 *J. Phys. A: Math. Gen.* **25** 1549
[21] Arratia O and del Olmo M A 1997 *Forschr. Phys.* **45** 103
[22] Santander M, Herranz F J and del Olmo M A 1992 Kinematics and homogeneous spaces for symmetrical

contractions of orthogonal groups *Anales de Física*, *Monografías 1* vol I, ed M A del Olmo, M Santander and J Mateos-Guilarte (Madrid: CIEMAT/RSEF) p 455

[23] Gauntlett J P, Gomis J and Townsend P K 1990 *Phys. Lett.* B **248** 288

[24] Cornwell J F 1989 *Group Theory in Physics* vol III (London: Academic)

[25] de Azcárraga J A and Ginestar D 1991 *J. Math. Phys.* **32** 3500

[26] Puzalowski R 1978 *Acta Phys. Austr.* **50** 45